COMMON BOUNDARY VALUES OF HOLOMORPHIC FUNCTIONS FOR TWO-SIDED COMPLEX STRUCTURES

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Abstract. Let Ω_1, Ω_2 be two disjoint open sets in $\mathbf{C}^n$ whose boundaries share a smooth real hypersurface M as relatively open subsets. Assume that Ω_i is equipped with a complex structure J^i which is smooth up to M. Assume that the operator norm $\|J^2 - J^1\| < 2$ on M. Let f be a continuous function on the union of Ω_1, Ω_2, M. If f is holomorphic with respect to both structures in the open sets, then f must be smooth on the union of Ω_1 with M. Although the result as stated is far more meaningful for integrable structures, our methods make it much more natural to deal with the general almost complex structures without the integrability condition. The result is therefore proved in the framework of almost complex structures.

Contents

1. Introduction

The title of the paper is suggested by the following result.

Proposition 1.1. *Let $0 < \alpha < 1$ and $k \geq 0$ be an integer. Let γ be an embedded curve in $\mathbf{C}$ of class $\mathcal{C}^{k+1+\alpha}$. Let Ω_1, Ω_2 be disjoint open subsets of $\mathbf{C}$. Suppose that both boundaries $\partial\Omega_1$, $\partial\Omega_2$ contain γ as relatively open subsets. Assume that $a_i \in \mathcal{C}^{k+\alpha}(\Omega_i \cup \gamma)$ satisfy $|a_i(z)| < 1$ on $\Omega_i \cup \gamma$. Let f be a continuous function on $\Omega_1 \cup \gamma \cup \Omega_2$ satisfying*

$$\partial_{\overline{z}} f + a_i \partial_z f = 0 \quad \text{on } \Omega_i, \quad i = 1, 2.$$

Then f is in $\mathcal{C}^{k+1+\alpha}_{loc}(\Omega_1 \cup \gamma) \cap \mathcal{C}^{k+1+\alpha}_{loc}(\Omega_2 \cup \gamma)$.

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Here $\mathcal{C}^{k+\alpha}_{loc}(\Omega\cup\gamma)$ denotes the set of functions $f\in\mathcal{C}^k(\Omega\cup\gamma)$ whose k-th order derivatives are in $\mathcal{C}^\alpha(K)$ for each compact subset of $\Omega\cup\gamma$. The result does not hold in general for harmonic functions due to a jump formula for single-layer distributions. A more general result for non-homogeneous equations is in Proposition 6.2. Our next result is in higher dimension.

Theorem 1.2. *Let $k\geq 4, n\geq 2$ be integers and let $0<\alpha<1$. Let Ω_1,Ω_2 be disjoint open subsets of $\mathbf{C}^n$ such that both boundaries $\partial\Omega_1$, $\partial\Omega_2$ contain a smooth real hypersurface M of class $\mathcal{C}^{k+1+\alpha}$ as relatively open subsets. For $i=1,2$, let J^i be an almost complex structure of class $\mathcal{C}^{k+\alpha}$ on $\Omega_i\cup M$. Suppose that on M the operator norm $\|J^2-J^1\|<2$. Let f be a continuous function on $\Omega_1\cup M\cup\Omega_2$. Suppose that for $i=1,2$, $1\leq j\leq n$, $(\partial_{x_j}+\sqrt{-1}J^i\partial_{x_j})f$ and $(\partial_{y_j}+\sqrt{-1}J^i\partial_{y_j})f$, defined on Ω_i, extend to functions in $\mathcal{C}^k(\Omega_i\cup M)$. Then f is of class $\mathcal{C}^{k-3+\beta}(\Omega_1\cup M)$ for all $\beta<1$. In particular, $f\in\mathcal{C}^\infty(\Omega_1\cup M)$ when $k=\infty$.*

Notice that no assumption is made on convexity of M with respect to either of the almost complex structures. The definition of almost complex structures is in section 3 and a general result is in section 5.

We would like to mention that the interior regularity of f for integrable almost complex structures is ensured by the well-known Newlander-Nirenberg theorem [9] (see also Nijenhuis-Woolf [10] and Webster [14]). There are results on Newlander-Nirenberg theorem for pseudoconvex domains with boundary by Catlin [2] and Hanges-Jacobowitz [5]. See earlier work of Hill [6] on failure of Newlander-Nirenberg type theorem with boundary.

We now observe how the common boundary values arise in the Cauchy-Green operator for $\overline{\partial}$ in $\mathbf{C}$. Let Ω be a bounded domain in $\mathbf{C}$ with $\partial\Omega\in\mathcal{C}^{k+1+\alpha}$. Seeking coordinates $z+f(z)$ to transform $\partial_{\overline{z}}+a\partial_z$ into a multiple of $\partial_{\overline{z}}$ leads to the equation

$$\partial_{\overline{z}}f+a(z)\partial_z f+b(z)=0,\quad z\in\Omega, \tag{1.1}$$

where $a\in\mathcal{C}^{k+\alpha}(\overline{\Omega})$, and b is either a or a function of the same kind. To solve it, one considers the integro-differential equation

$$f(z)+T(a\partial_z f)(z)+Tb(z)=0,\quad z\in\Omega. \tag{1.2}$$

Here $T=T_\Omega$ is the Cauchy-Green operator

$$Tf(z)=\frac{1}{\pi}\int_\Omega\frac{f(\zeta)}{z-\zeta}\,d\xi\,d\eta.$$

The equation (1.2) is equivalent to (1.1) and an extra equation

$$\int_{\partial\Omega}\frac{f(\zeta)}{\zeta-z}\,d\zeta=0,\quad z\in\Omega. \tag{1.3}$$

When $f\in\mathcal{C}^0(\partial\Omega)$, the jump formula implies that (1.3) is equivalent to f being the boundary value of a function which is holomorphic on $\Omega'=\mathbf{C}\setminus\overline{\Omega}$, continuous on $\overline{\Omega'}$, and vanishing at ∞. (See Lemma 6.4 for details.)

As an application of Proposition 1.1, we will prove the following.

Theorem 1.3. *Let $0<\alpha<1$ and let $\Omega\subset\mathbf{C}$ be a bounded domain with $\mathcal{C}^{1+\alpha}$ boundary. Let $a,b\in\mathcal{C}^\alpha(\overline{\Omega})$. There exists $\epsilon_\alpha>0$ such that if $\|a\|_\alpha<\epsilon_\alpha$, then (1.2) admits a unique*

solution $f \in \mathcal{C}^{1+\alpha}(\overline{\Omega})$. *Assume further that* $a, b \in \mathcal{C}^{k+\alpha}(\overline{\Omega})$ *and* $\partial\Omega \in \mathcal{C}^{k+1+\alpha}$ *for an integer* $k \geq 0$. *Then* $f \in \mathcal{C}^{k+1+\alpha}(\overline{\Omega})$. *Consequently, the linear map* $\mathrm{I} + Ta\partial_z$ *from* $\mathcal{C}^{k+1+\alpha}(\overline{\Omega})$ *into itself has a bounded inverse.*

The structures $\partial_{\overline{z}}$ and ∂_z show that Theorem 1.2 fails for $\|J^2 - J^1\| = 2$ and $M\colon\ \mathrm{Im}\, z_1 = 0$. We expect that the regularity of f in Theorem 1.2 can be improved. The loss of derivatives is due to an essential use of the Fourier transform. For this reason, we will present two alternative proofs for the one-dimensional case, with one producing the sharp result.

We want to mention two open problems in addition to the regularity issue mentioned above. The first problem is on the vector-valued version of (1.2). The second is concerned with non-linear integro-differential equations arising from differential equations of J-holomorphic curves; in fact, the integrable case remains to be studied.

Problem A. *Let* $m \geq 2, 0 < \alpha < 1$ *and let* D *be a bounded domain in* $\mathbf{C}$ *with* $\mathcal{C}^\infty$ *boundary. Let* $A = (a_{jk}) \in \mathcal{C}^\infty(\overline{D})$ *be an* $m \times m$ *matrix with sufficiently small* $\mathcal{C}^\alpha$ *norm on* $\overline{D}$. *Does* $\mathrm{I} + T_D A\partial_z\colon [\mathcal{C}^{k+\alpha}(\overline{D})]^m \to [\mathcal{C}^{k+\alpha}(\overline{D})]^m$ *have a bounded inverse for all positive integer* k*?*

Problem B. *Let* D *be a bounded domain in* $\mathbf{C}$ *with* $\mathcal{C}^\infty$ *boundary. Let* Ω *be a domain in* $\mathbf{C}^n$ *with* $n \geq 1$. *Let* A *be an* $n \times n$ *matrix of* $\mathcal{C}^\infty$ *functions on* Ω. *Suppose that the operator norm* $\|A(z)\|$ *is less than* 1 *on* Ω. *Suppose that a continuous map* $u\colon \overline{D} \to \Omega$ *satisfies*

$$u + T_D(A(u)\overline{\partial_z u}) = T_D v \tag{1.4}$$

on D. *Here* v *is a* $\mathcal{C}^\infty$ *map from* $\overline{D}$ *into* $\mathbf{C}^n$. *Is* $u \in \mathcal{C}^\infty(\overline{D})$*?*

Note that the interior regularity of u is in work of Nijenhuis-Woolf [10]. When the $\mathcal{C}^{1+\alpha}$ norm of A is sufficiently small, the existence and uniqueness of solutions u to (1.4) is ensured. Problems A and B can be reformulated in terms of two differential equations on D and its complement. Indeed, by Lemma 6.4 a continuous map $u\colon \overline{D} \to \Omega$ satisfies (1.4) if and only if it extends to a continuous map u from $\mathbf{C}$ into itself that vanishes at ∞ and satisfies $\partial_{\overline{z}} u + A(u)\overline{\partial_z u} = v$ on D and $\partial_{\overline{z}} u = 0$ on $\mathbf{C} \setminus \overline{D}$.

2. Inverting $\mathrm{I} + TA\overline{\partial_z}$

In this section, we will recall estimates on the Cauchy-Green operator T and $\partial_z T$. We will discuss the inversion of $\mathrm{I} + TA\partial_z$ in spaces of higher order derivatives when A has a small $\mathcal{C}^\alpha$ norm. When A has compact support, it is easy to bound inverses of $\mathrm{I} + TA\partial_z, \mathrm{I} + TA\overline{\partial_z}$. We will show in section 6 that $\mathrm{I} + TA\partial_z$ is indeed invertible when A is a suitable scalar function.

Throughout the paper, when a parameter set P is involved in $\Omega \times P$, Ω is a bounded open set in a euclidean space and P is the closure of a bounded open set in a euclidean space. We assume that two points a, b in $\overline{\Omega} \times P$ can be connected by a smooth curve in $\overline{\Omega} \times P$ of length at most $C|b - a|$.

We will need spaces of functions with parameter. The usual norm on $\mathcal{C}^{k+\alpha}(\overline{\Omega} \times P)$ is denoted by $|\cdot|_{k+\alpha}$. Following [10], for integers $k, j \geq 0$ we define $\hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega}, P)$ to be the set of functions $f(z,t)$ such that for all $i \leq j$, $\partial_t^i f \in \mathcal{C}^k(\overline{\Omega} \times P)$ and

$$\|f\|_{k+\alpha,j} = \max_{0 \leq i \leq j} |\partial_t^i f(\cdot, t)|_{k+\alpha} < \infty.$$

Define $\hat{\mathcal{C}}^{\infty,j}(\overline{\Omega},P)=\bigcap_{k=1}^{\infty}\hat{\mathcal{C}}^{k,j}(\overline{\Omega},P)$. Throughout the paper, k is a nonnegative integer, and $0<\alpha<1$. To simplify notation, the parameter set P will not be indicated sometimes.

Let Ω be a bounded domain in $\mathbf{C}$. The $\overline{\partial}$ solution operator T and $S=\partial_z T$ are

$$Tf(z)=\frac{1}{\pi}\int_{\Omega}\frac{f(\zeta)}{z-\zeta}\,d\xi\,d\eta,\quad Sf(z)=-\frac{1}{\pi}\,p.v.\int_{\Omega}\frac{f(\zeta)}{(z-\zeta)^2}\,d\xi\,d\eta. \tag{2.1}$$

It is well-known that $\partial_{\overline{z}}T$ is the identity on $L^p(\Omega)$ when $p>2$. When $f\in\mathcal{C}^{\alpha}(\overline{D})$ and $\partial\Omega\in\mathcal{C}^{1+\alpha}$, one has

$$Sf(z)=-\frac{1}{\pi}\int_{\Omega}\frac{f(\zeta)-f(z)}{(z-\zeta)^2}\,d\xi\,d\eta-\frac{f(z)}{2\pi i}\int_{\partial\Omega}\frac{d\overline{\zeta}}{\zeta-z}. \tag{2.2}$$

If f has compact support in Ω, or if $f\in\mathcal{C}^{k+\alpha}(\overline{\Omega})$ and $\partial\Omega\in\mathcal{C}^{k+1+\alpha}$, then T, S satisfy

$$|Tf|_{k+1+\alpha}\leq C_{k+1+\alpha}|f|_{k+\alpha},\quad |Sf|_{k+\alpha}\leq C_{k+1+\alpha}|f|_{k+\alpha}. \tag{2.3}$$

See Bers [1] and Vekua [13] (p. 56). The above estimates for domains with parameter will be derived in section 4. It is known that

$$\partial_z Sf=S\partial_z f,\quad \partial_{\overline{z}}Sf=\partial_z f, \tag{2.4}$$

where the first identity needs f to have compact support in Ω.

For $f\in\hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega},P)$, define $Tf(z,t), Sf(z,t)$ by (2.1)-(2.2) by fixing t.

Lemma 2.1. *Let $\Omega\subset\mathbf{C}$ be a bounded domain with $\partial\Omega\in\mathcal{C}^{k+1+\alpha}$. Then*

$$T\colon\hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega},P)\to\hat{\mathcal{C}}^{k+1+\alpha,j}(\overline{\Omega},P),\quad \|Tf\|_{k+1+\alpha,j}\leq C_{k+1+\alpha}\|f\|_{k+\alpha,j},$$
$$S\colon\hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega},P)\to\hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega},P),\quad \|Sf\|_{k+\alpha,j}\leq C_{k+1+\alpha}\|f\|_{k+\alpha,j}. \tag{2.5}$$

Proof. By (2.2), we get $S(\hat{\mathcal{C}}^{k+\alpha,j})\subset\hat{\mathcal{C}}^{k,j}$. We can verify that $\partial_t S=S\partial_t$ on $\hat{\mathcal{C}}^{\alpha,j}$ for $j\geq 1$. Thus $S(\hat{\mathcal{C}}^{k+\alpha,j})\subset\hat{\mathcal{C}}^{k+\alpha,j}$ by (2.3).

The Cauchy kernel is integrable. So $T(\hat{\mathcal{C}}^{0,j}(\overline{\Omega}\times P))\subset\hat{\mathcal{C}}^{0,j}(\overline{\Omega}\times P)$. Also $\partial_t T=T\partial_t$ on $\hat{\mathcal{C}}^{0,j}$ for $j\geq 1$. The rest of assertions follows from $\partial_z T=S$ and $\partial_{\overline{z}}T=\mathrm{I}$. □

By an abuse of notation, we define $\overline{\partial_z}f=\overline{\partial_z f}$.

Lemma 2.2. *Let Ω be a bounded domain in $\mathbf{C}$. Let $A\in\hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega},P)$ be an $m\times m$ matrix. There exists ϵ_α which depends only on α and satisfies the following.*

(i) If $\partial\Omega\in\mathcal{C}^{1+\alpha}$ and $|A|_{\alpha,0}<\epsilon_\alpha$, then

$$\mathrm{I}+TA\partial_z,\ \mathrm{I}+TA\overline{\partial_z}\colon\left[\hat{\mathcal{C}}^{1+\alpha,j}(\overline{\Omega})\right]^m\to\left[\hat{\mathcal{C}}^{1+\alpha,j}(\overline{\Omega})\right]^m$$

have bounded inverses.

(ii) If $A(\cdot,t)$ have compact support in Ω for all $t\in P$ and $|A|_{\alpha,0}<\epsilon_\alpha$, then

$$\mathrm{I}+TA\partial_z,\ \mathrm{I}+TA\overline{\partial_z}\colon\left[\hat{\mathcal{C}}^{k+1+\alpha,j}(\overline{\Omega})\right]^m\to\left[\hat{\mathcal{C}}^{k+1+\alpha,j}(\overline{\Omega})\right]^m$$

have bounded inverse.

Proof. To be made precise, when S operates on functions with compact support, it commutes with $\partial_t, \partial_z, \partial_{\overline{z}}$ somewhat. However, differentiating the operator product $(SA)^n$ requires counting terms efficiently as n tends to ∞.

(i). Fix $0 < \theta < 1/2$. Note that

$$\|fg\|_{k+\alpha,j} \leq C_{k,j}\|f\|_{k+\alpha,j}\|g\|_{k+\alpha,j}.$$

By (2.5), we have $\|SA\|_{\alpha,0} \leq C'_\alpha\|A\|_{\alpha,0}$. Thus,

$$\|(SA)^n\|_{\alpha,0} \leq (C_\alpha\|A\|_{\alpha,0})^n \leq \theta^n$$

if $|A|_{\alpha,0}$ is sufficiently small. Note that

$$(TA\partial_z)^n = TA(SA)^{n-1}\partial_z.$$

Let $L = \mathrm{I} + \sum_{n=1}^{\infty}(-1)^n TA(SA)^{n-1}\partial_z$. Then

$$\|TA(SA)^{n-1}\partial_z f\|_{1+\alpha,0} \leq C_\alpha\theta^{n-1}\|A\|_{\alpha,0}\|f\|_{1+\alpha,0}.$$

This shows that for $f \in \hat{\mathcal{C}}^{1+\alpha,0}$, $\sum_{n=0}^{\infty}(-1)^n TA(SA)^{n-1}\partial_z f$ converges to $Lf \in \hat{\mathcal{C}}^{1+\alpha,0}$. Moreover, $\|Lf\|_{1+\alpha,0} \leq C\|f\|_{1+\alpha,0}$. It is straightforward that $L(\mathrm{I} + TA\partial_z)$ and $(\mathrm{I} + TA\partial_z)L$ are the identity on $\hat{\mathcal{C}}^{1+\alpha,0}$. This verifies (i) for $j = 0$. The case of $j > 0$ will follow from the argument in (ii) below, by using $\partial_t T = T\partial_t$, $\partial_t S = S\partial_t$.

(ii). We need to show that $\sum\|(SA)^n\|_{k+\alpha,j}$ converges when A has compact support in Ω. Denote by $C_{k+\alpha,j}$ a constant depending only on k, j, and $\|A\|_{k+\alpha,j}$. By (2.4) and $\partial_t S = S\partial_t$, we can write

$$\partial SA = \tilde{S}\tilde{\partial}A.$$

Here $\tilde{S}$ is either S or I, and $\partial, \tilde{\partial}$ are of form $\partial_z, \partial_{\overline{z}}, \partial_t$. Denote by ∂^K derivatives in z, $\overline{z}$. Then $\partial(SA)^n$ equals a sum of terms of the form

$$S_{m_1}(\partial^{K_1}A)\cdots S_{m_n}(\partial^{K_n}A)\partial^{K_{n+1}}, \quad |K_1| + \cdots + |K_{n+1}| = 1.$$

Here S_{m_i} is either S or I; in particular, $\|S_{m_i}\|_{k+\alpha,j} \leq C_{k+\alpha,j}$ for all m_i. The sum has at most $n+1$ terms. Thus $\partial^K\partial_t^J(SA)^n$ is a sum of at most $(n+1)^{|K|+|J|}$ terms of

$$S_{m_1}(\partial^{K_1}\partial_t^{J_1}A)\cdots S_{m_n}(\partial^{K_n}\partial_t^{J_n}A)S_{m_n}\partial^{K_{n+1}}\partial_t^{J_{n+1}}. \tag{2.6}$$

Assume that $|K| \leq k, |J| \leq j$ and $n > k+j$. With $C_\alpha \geq 1$,

$$\begin{aligned}|\partial^K\partial_t^J((SA)^n f)|_{\alpha,0} &\leq (n+1)^{k+j}C_\alpha^n(1+|A|_{k+\alpha,j})^{k+j}|A|_{\alpha,0}^{n-k-j}\|f\|_{k+\alpha,j}\\ &\leq (n+1)^{k+j}C_{k+\alpha,j}\|f\|_{k+\alpha,j}\theta^{n-k-j}.\end{aligned}$$

This shows that $\|(SA)^n\|_{k+\alpha,j} \leq C'_{k+\alpha,j}(n+1)^{k+j}\theta^{n-k-j}$. Hence

$$\|TA(SA)^n\partial_z\|_{k+1+\alpha,j} \leq C_{k+\alpha,j}(n+1)^{k+j}\theta^{n-k-j}.$$

We conclude that $\|(\mathrm{I} + TA\partial_z)^{-1}\|_{k+1+\alpha,j} < \infty$.

The proof for $\mathrm{I} + TA\overline{\partial_z}$ is obtained by minor changes. Indeed, with $Cf = \overline{f}$, we write $(TA\overline{\partial_z})^n = TAC(SAC)^{n-1}\partial_z$. Now, $\partial_z C = C\partial_{\overline{z}}$ and $\partial_{\overline{z}}C = C\partial_z$. We may assume that t are real variables. So $\partial_t C = C\partial_t$. Thus $\partial^K\partial_t^J(SAC)^n$ is a sum of at most $(n+1)^{|K|+|J|}$ terms of

$$S_{m_1}(\partial^{K_1}\partial_t^{J_1}A)C\cdots S_{m_n}(\partial^{K_n}\partial_t^{J_n}A)C\partial^{K_{n+1}}\partial_t^{J_{n+1}}.$$

Substitute the above for (2.6). The remaining argument follows easily. $\square$

We need a simple version of Whitney's extension theorem with parameter.

Lemma 2.3. *Let N be a positive integer or ∞, $0 \leq j < \infty$, and $0 \leq \alpha < 1$. Let e_k be a sequence of positive numbers. Let $f_I \in \hat{\mathcal{C}}^{N-1-|I|+\alpha,j}(\mathbf{R}^n, P)$ for $0 \leq |I| < N$ with $I = (i_1, \ldots, i_m)$. Assume that all $f_I(\cdot, t)$ have support contained in a compact subset K of the unit ball B. There exists $Ef \in \hat{\mathcal{C}}^{N-1+\alpha,j}(\mathbf{R}^n \times \mathbf{R}^m, P)$ such that $\partial_y^I Ef(x,0,t) = f_I(x,t)$. Moreover, $Ef(\cdot, t)$ have compact support in the unit ball of $\mathbf{R}^n \times \mathbf{R}^m$ and*

$$\|Ef\|_{k+\alpha,j} \leq \epsilon_k + C_{N,k,K} \sum_{|I|\leq k} \|f_I\|_{k-|I|+\alpha,j}, \quad 0 \leq k < N. \tag{2.7}$$

Here $E, C_{N,k,K}$ are independent of j.

Proof. When $N = \infty$, take $g \in \mathcal{C}_0^\infty(\Delta_\delta^m)$ with $g(y) - 1$ vanishing to infinity order at 0. Set

$$Ef(x,y,t) = \sum \frac{y^I}{I!} f_I(x,t) g(\delta_{|I|}^{-1} y). \tag{2.8}$$

For the above to converge, choose δ_i which decrease to 0 so rapidly that for all t

$$\|b_I\|_{i-1,j} \leq \delta_i^{1/2}, \quad b_I(x,y,t) = y^I g(\delta_i^{-1} y) f_I(x,t), \quad i = |I| \geq 1.$$

Thus, $Ef \in \hat{\mathcal{C}}^{k,j}$ for all k. Note that we can choose δ_k so small that $\|Ef\|_{k,j} \leq e_k + C_k \sum_{|I|\leq k} \|f_I\|_{k,j}$ for all k.

Let $N < \infty$. Let ϕ be a smooth function on $\mathbf{R}^n$ with support in B_δ^n for a small δ. Also $\int_{\mathbf{R}} \phi(y)\, dy = 1$. We extend f one dimension at a time. Assume that $m = 1$. We need to modify extension (2.8). We replace $y^i f_i(x,t)$ by $y^i g_i(x,y,t)$ to achieve the $\hat{\mathcal{C}}^{N-1+\alpha,j}$ smoothness. We also need the correct i-th y-derivative of $y^i g_i(x,y,t)$ due to the presence of $y^l g_l(x,y,t)$ for $l < i$. With $a_i \in \hat{\mathcal{C}}^{N-1-i,j}$ to be determined, consider

$$g_i(x,y,t) = \int_{\mathbf{R}^n} a_i(x - yz, t)\phi(z)\, dz.$$

Fix t. We first show that $y^i g_i(x,y,t)$ is of class $\mathcal{C}^{N-1+\alpha,j}$. Since it is $\mathcal{C}^\infty$ for $y \neq 0$, it suffices to extend its partial derivative of order $< N$ on $y \neq 0$ continuously to $\mathbf{R}^n \times \mathbf{R}$, as the extensions are clearly independent of the order of differentiation. By the product rule, this amounts to extending $y^{i-l}\partial^I g_i$ for $|I| \leq N-1-l$. When $|I| \leq N-1-i$, we take $I_1 = I$ and $I_2 = 0$. Otherwise, $I = I_1 + I_2$ with $|I_1| = N-1-i$. Then

$$\partial^{I_1} g_i(x,y,t) = \sum_{|L|=|I_1|} \int \partial^L a_i(x - yz, t)\phi_{I_1 L}(z)\, dz$$

for some $\phi_{I_1 L}$ with support in B_δ^n. When $y \neq 0$, change variables and take derivative ∂^{I_2}. We get

$$\partial^I g_i(x,y,t) = \sum_{|L|=|I_1|} \int \frac{1}{y^{|I_2|+n}} \partial^L a_i(z) \tilde{\phi}_{I_1 I_2 L}\left(\frac{x-z}{y}\right) dz.$$

Change variables again. We get

$$y^{|I_2|}\partial^I g_i(x,y,t) = \sum_{|L|=|I_1|} \int \partial^L a_i(x - yz, t)\tilde{\phi}_{I_1 I_2 L}(z)\, dz.$$

The right-hand side and its derivatives in t of order at most j are clearly continuous functions. Since $|I_2| \leq (N-1-l)-(N-1-i) = i-l$, then $y^{i-l}\partial^I g_i$ extends continuously to $\mathbf{R}^n \times \mathbf{R}$. Take derivative in parameter t and compute the Hölder ratio in x, y. We get

$$\|b_{i,lI}\|_{\alpha,j} \leq C_{N,K}\|a_i\|_{N-1-i+\alpha,j}, \quad b_{i,lI}(x,y,t) = y^{i-l}\partial^I g_i(x,y,t).$$

By the product rule, at $y = 0$

$$\partial_y^i(y^i g_i(x,y,t)) = i!a_i(x,t), \quad \partial_y^l(y^i g_i(x,y,t)) = 0, \quad l < i.$$

Starting with $a_0 = f_0$, inductively we find $a_i \in \hat{\mathcal{C}}^{N-1-i+\alpha,j}$ such that

$$Ef(x,y,t) = \sum_{i<N} \frac{y^i}{i!} g_i(x,t) g(\delta^{-1}y)$$

satisfies $\partial_y^i Ef(x,y,t) = f_i(x,t)$ for $i = 0, 1, \ldots, N-1$. The estimate (2.7) is immediate when δ is sufficiently small.

For $m > 1$, suppose that we have found extensions $\tilde{f}_i \in \hat{\mathcal{C}}^{N-1-i+\alpha,j}(\mathbf{R}^n \times \mathbf{R}^{m-1}, P)$ such that $\partial_{y'}^{I'}\tilde{f}_i = f_{I'i}$ at $y' = 0$ for all $|I'| < N-i$ and

$$\|\tilde{f}_i\|_{k-i+\alpha,j} \leq e'_{N,K} + C_{N,K} \sum_{|I'| \leq k-i} \|f_{I'i}\|_{k-|I'|-i+\alpha,j}, \quad k < N \tag{2.9}$$

with $e'_{N,K} > 0$ to be determined. Assume further that $\tilde{f}_i(\cdot,t)$ have support in a compact subset K' of the unit ball of $\mathbf{R}^{n+m-1}$, where K' depends only on K. Using the one-dimensional result again, we get $Ef \in \hat{\mathcal{C}}^{N-1+\alpha,j}(\mathbf{R}^n \times \mathbf{R}^m, P)$ with compact support in the unit ball of $\mathbf{R}^{n+m}$. Furthermore, $\partial_{y_n}^i Ef = \tilde{f}_i$ at $y_n = 0$ and

$$\|Ef\|_{k+\alpha,j} \leq e'_{N,K} + C'_{N,K} \sum_{0 \leq i \leq k} \|\tilde{f}_i\|_{k-i+\alpha,j}, \quad k < N.$$

Let $e'_{N,K}$ be sufficiently small. Combining with (2.9) yields (2.7). $\square$

The above proof for non-parameter case is in [7] (pp. 16 and 18). When f is defined on $y_n \leq 0$ with $\partial_{x_n}^k f = f_k$ on $y_n = 0$, the above extension Ef can be replaced by f on $y_n \leq 0$. The same conclusions on Ef hold. Seeley [12] has a linear extension $E\colon \mathcal{C}^\infty(\overline{\mathbf{R}}^n_+) \to \mathcal{C}^\infty(\mathbf{R}^n)$ such that $E\colon \mathcal{C}^k(\overline{\mathbf{R}}^n_+) \to \mathcal{C}^k(\mathbf{R}^n)$ have bounds depending only on k.

3. J-holomorphic curves and derivatives on curves

In this section, we first explain how we arrive at the condition $\|J_i - J_{st}\| < 2$ in Theorem 1.2. Our second result is about J-holomorphic curves with parameter. The result is essentially in work of Nijenhuis-Woolf [10]. See also Ivashkovich-Rosay [8] for another regularity proof and jets of J-holomorphic curves. The proof below relies only on some basic facts about the Cauchy-Green operator and the inversion of $\mathrm{I} + TA\overline{\partial_z}$ discussed in section 2. Finally, we will express partial derivatives through a family of derivatives on curves.

Our results are local. Throughout the paper, a real hypersurface M will be a relatively open subset of the boundary of a domain in $\mathbf{C}^n$, or a closed subset without boundary in the domain.

Let Ω be a domain in $\mathbf{R}^{2n}$ and M be a (relatively open) subset of $\partial\Omega$. Let $k \geq 1$. We say that $X_1, \ldots, X_n$ define an *almost complex structure* J on Ω (resp. $\Omega \cup M$) of class $\mathcal{C}^{k+\alpha}$, if X_j's and their conjugates are pointwise $\mathbf{C}$-linearly independent on Ω (resp. $\Omega \cup M$) and X_j are of class $\mathcal{C}^{k+\alpha}$ on Ω (resp. $\Omega \cup M$). Note that J_p is defined to be the linear map on $T_p\Omega$ (resp. $T_p(\Omega \cup M)$) such that $v + \sqrt{-1}J_p v$ is in the linear span of $X_1(p), \ldots, X_n(p)$. The *operator norm* of a linear map A from $T_p(\Omega)$ into itself is defined as $\max\{\|Av\| \colon \|v\| = 1\}$ with $\|\cdot\|$ being the euclidean norm on $T_p\Omega \equiv \mathbf{R}^{2n}$. We say that a diffeomorphism φ transforms $X_1, \ldots, X_n$ into $\tilde{X}_1, \ldots, \tilde{X}_n$, if $d\varphi(X_j)$ are locally in the span of $\tilde{X}_1, \ldots, \tilde{X}_n$.

A linear complex structure J on $\mathbf{C}^n$ is given by

$$X_j = \sum_{1 \leq k \leq n} (b_{jk}\partial_{\overline{z}_k} + a_{jk}\partial_{z_k}), \quad 1 \leq j \leq n,$$

where constant matrices $A = (a_{jk})$ and $B = (b_{jk})$ satisfy

$$\begin{vmatrix} B & A \\ \overline{A} & \overline{B} \end{vmatrix} \neq 0.$$

Denote by A^t the transpose matrix of A. The map $z = \overline{B}^t w + A^t \overline{w}$ transforms $\partial_{\overline{w}_1}, \ldots, \partial_{\overline{w}_n}$ into $X_1, \ldots, X_n$, and hence J_{st} into J given by

$$J = (K^t)^{-1} J_{st} K^t, \quad J_{st} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}, \quad K = \begin{pmatrix} \mathrm{Re}\,(B+A) & \mathrm{Im}\,(A+B) \\ \mathrm{Im}\,(A-B) & \mathrm{Re}\,(B-A) \end{pmatrix}. \tag{3.1}$$

Thus under a local change of coordinates by shrinking Ω or $\Omega \cup M$, an almost complex structure J is given by

$$X_j = \partial_{\overline{z}_j} + \sum_{1 \leq k \leq n} a_{jk}(z)\partial_{z_k}, \quad j = 1, \ldots, n \tag{3.2}$$

with operator norm $\|(a_{jk})(z)\| < 1$ on Ω (resp. $\Omega \cup M$).

Lemma 3.1. *Let J^1, J^2 be two linear complex structures on $\mathbf{R}^{2n}$. Let M be a hyperplane in $\mathbf{R}^{2n}$. There exists $v \in T_0M$ such that J^1v, J^2v are in the same connected component of $T_0\mathbf{R}^{2n} \setminus T_0M$, provided $T_0M \cap J^1T_0M \neq T_0M \cap J^2T_0M$, or the operator norm $\|J^2 - J^1\| < 2$.*

Proof. To simplify notations, all tangent vectors or spaces are at the origin. Let $T(M, J^i) = TM \cap J^iTM$. Let ω_1, ω_2 be two connected components of $T\mathbf{R}^{2n} \setminus TM$. Note that J^i sends one of two connected components of $TM \setminus T(M, J^i)$ into ω_1 and the other into ω_2. Thus the assertion is trivial, if $T(M, J^1) \neq T(M, J^2)$. Assume that they are identical.

By choosing an orthonormal basis for $T\mathbf{R}^{2n}$, we may assume that $T(M, J^1)$ is given by $x_n = y_n = 0$. Since M contains $x_n = y_n = 0$, then M is defined by $y_n' = ax_n + by_n = 0$ with $a^2 + b^2 = 1$. By a change of orthonormal coordinates, M, $T(M, J^1)$ are defined by $y_n = 0$ and $x_n = y_n = 0$ respectively. Write

$$J^i \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix} = A_i \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix}, \quad J^i \begin{pmatrix} \partial_{x_n} \\ \partial_{y_n} \end{pmatrix} = C_i \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix} + D_i \begin{pmatrix} \partial_{x_n} \\ \partial_{y_n} \end{pmatrix}.$$

Here A_i, C_i, D_i are matrices. In particular, $D_i^2 = -\mathrm{I}$. We want to show that the coefficients of ∂_{y_n} in $J^i\partial_{x_n}$ have the same sign. Otherwise, we can write

$$D_1 = \begin{pmatrix} a_1 & b_1 \\ -\frac{1+a_1^2}{b_1} & -a_1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} a_2 & -b_2 \\ \frac{1+a_2^2}{b_2} & -a_2 \end{pmatrix}, \quad b_1 > 0, \quad b_2 > 0.$$

We have $\|D_2 - D_1\| < 2$. Thus, $b_1 + b_2 < 2$ and $b_1^{-1} + b_2^{-1} < 2$, a contradiction. □

Example 3.2. Lemma 3.1 and Theorem 1.2 fail easily for the triplet $\{J_{st}, -J_{st}, \{y_1 = 0\}\}$. A less simple example is in higher dimension. Let $0 \leq t \leq \pi$, and let J_t be defined by

$$X_1^t = (\cos t\,\partial_{x_1} + \sin t\,\partial_{x_2}) + i\partial_{y_1}, \quad X_2^t = (-\sin t\,\partial_{x_1} + \cot t\,\partial_{x_2}) + i\partial_{y_2}.$$

Lemma 3.1 and Theorem 1.2 fail for $\{J_0, J_\pi, \{y_2 = 0\}\}$ with $\|J_0 - J_\pi\| = 2$. Under new orthonormal coordinates $w_1 = (x_2 + iy_1)/\sqrt{2}$, $w_2 = (-x_1 + iy_2)/\sqrt{2}$, J_t is given by

$$(1+\sin t)\partial_{\overline{w}_1} - \cos t\partial_{\overline{w}_2} - (1-\sin t)\partial_{w_1} - \cos t\,\partial_{w_2},$$
$$\cos t\,\partial_{\overline{w}_1} + (1+\sin t)\partial_{\overline{w}_2} + \cos t\,\partial_{w_1} - (1-\sin t)\partial_{w_2}.$$

The above can be put into (3.2) with

$$(a_{jk}^t) = \begin{pmatrix} 0 & -\frac{\cos t}{1+\sin t} \\ \frac{\cos t}{1+\sin t} & 0 \end{pmatrix}.$$

Note that $\|(a_{jk}^t)\| \leq 1$. However, we do not know if Theorem 1.2 holds for two structures of the form (3.2) with $\|(a_{jk})\| < 1$.

Let J be an almost complex structure defined by vector fields $X_1, \dots, X_n$ of class $\mathcal{C}^k$ on Ω with $k \geq 1$. A $\mathcal{C}^1$ map $u \colon \overline{\mathbb{D}}^+ \to \Omega$ is called an *approximate* J-holomorphic curve attached to the curve $u(x,0)$, if

$$du(\partial_{\overline{z}}) = D(z) \cdot X(u(z)) + F(z) \cdot \overline{X(u(z))}, \quad |F(z)| = o(|\operatorname{Im} z|^{k-1}).$$

If $F = 0$ and $\overline{\mathbb{D}}^+$ is replaced by $\mathbb{D}$, u is called J-holomorphic. Note that if f is a function on Ω the above equation implies that

$$\partial_{\overline{z}}(f(u(z)) = D(z) \cdot (Xf)(u(z)) + F(z) \cdot (\overline{X}f)(u(z)).$$

When X, J are defined by (3.2) and u is J-holomorphic, the identity becomes

$$\partial_{\overline{z}}(f(u(z)) = (Xf)(u(z)) \cdot \overline{\partial_z u}.$$

Next two results deal with the existence of the two types of curves.

Lemma 3.3. *Let $m \geq 0$ be an integer or $m = \infty$. Let $0 \leq \alpha < 1$. Let J be an almost complex structure defined by vector fields*

$$X_j = \sum_{1 \leq k \leq n} b_{jk}\partial_{\overline{z}_k} + \sum_{1 \leq k \leq n} a_{jk}\partial_{z_k}, \quad j = 1, \dots, n.$$

Assume that $A = (a_{jk}), B = (b_{jk})$ are of class $\mathcal{C}^{m+\alpha}(\Omega)$. Assume that $u_0 \colon (-1,1) \times P \to K$ is of class $\hat{\mathcal{C}}^{l+1+\alpha,j}((-1,1),P)$, and K is a compact subset of Ω. Let $l \geq 0, j \geq 0$, $j + l \leq m$, and $0 < r < 1$. There exists a map $u \colon [-r,r] \times [-\delta,\delta] \times P \to \Omega$ of class $\hat{\mathcal{C}}^{l+1+\alpha,j}([-r,r] \times [-\delta,\delta], P)$ satisfying the following.

(i) $u(x,0,t) = u_0(x,t)$ *and*

$$du(\partial_{\overline{z}}) = D(z,t)\cdot X(u(z,t)) + F(z,t)\cdot \overline{X(u(z,t))}, \tag{3.3}$$

$$|F(z,t)| = o(|y|^l),\ \alpha = 0; \quad |F(z,t)| = O(|y|^{l+\alpha}),\ 0<\alpha<1. \tag{3.4}$$

(ii) *Let* $e_l > 0$. *On* $[-r,r]\times[-\delta,\delta]\times P$ *the norms of* u, D, F *satisfy*

$$\|u\|_{l+1+\alpha,j} + \|(D,F)\|_{l+\alpha,j} \leq C_l^* \max(\|u_0\|_{l+1+\alpha,j}, \|u_0\|_{l+1+\alpha,j}^{l+j+2}) + e_l.$$

Moreover, $C_0^*(1+\|u_0\|_{1,0})\delta > 1$ *and* C_l^* *depends on* K, Ω *and*

$$|A|_{l+\alpha}, \quad |B|_{l+\alpha}, \quad \inf_{\Omega}\begin{vmatrix} B & A \\ \overline{A} & \overline{B}\end{vmatrix}.$$

Proof. We suppress the parameter t in all expressions. We first determine a unique set of coefficients $a_1(x),\dots,a_{l+1}(x)$ such that as a power series in y, $u(x,y) = u_0(x) + \sum_{i=1}^{l+1} a_i(x)y^i$ satisfies (3.3)-(3.4). It is convenient to regard u as the real map $(x,y)\to(\operatorname{Re}u, \operatorname{Im}u)$, which is still denoted by u, and rewrite the equations as

$$du(\partial_y) = J(u)(du(\partial_x)) + F_1(x,y)\cdot\partial_*, \quad F_1(x,y) = o(|y|^l). \tag{3.5}$$

Here $\partial_* = (\partial_{u_1},\dots,\partial_{u_{2n}})$ is evaluated at $u(z,t)$. In the matrix form, let J be the matrix defined by (3.1). Then we need to solve

$$\partial_y u = \partial_x u J(u) + F_2(x,y), \quad |F_2(x,y)| = o(|y|^l).$$

We solve the equation formally, which determines $a_1(x),\dots,a_{l+1}(x)$ uniquely, and then apply the Whitney extension (Lemma 2.3). This gives us a map u from $([-r,r]\times[-1,1])\times P$ into $\mathbf{R}^n$ of class $\mathcal{C}^{l+1+\alpha,j}$ satisfying the stated norm estimate. By $|u(x,y)-u(x,0)| \leq C_0^*(\|u_0\|_{1,0}+e_1)|y|$ and the compactness of K, we find $\delta>0$ such that u maps $[-r,r]\times[-\delta,\delta]\times P$ into Ω. We have obtained (3.5). Thus

$$2du(\partial_{\overline{z}}) = du(\partial_x) + idu(\partial_y) = du(\partial_x) + iJ(u)(du(\partial_x)) + iF_1(x,y)\partial_*.$$

Note that $du(\partial_x) + iJ(u)(du(\partial_x)) = D_1(z)\cdot X(u(z))$. Write ∂_* in terms of $X_i, \overline{X}_j$ by using the inverse of $\begin{pmatrix} B & A \\ \overline{A} & \overline{B}\end{pmatrix}$. We get (3.3)-(3.4). We can estimate the norms of D, F via D_1, F_i and the inverse matrix. □

We have defined $\hat{\mathcal{C}}^{k+\alpha,j}$, $\|\cdot\|_{k+\alpha,j}$ in section 2. Following [10], we define for $j\leq k$

$$\mathcal{C}^{k+\alpha,j}(\overline{\Omega},P) = \bigcap_{0\leq l\leq j}\hat{\mathcal{C}}^{k-l+\alpha,l}(\overline{\Omega},P), \quad |u|_{k+\alpha,j} = \max_{0\leq l\leq j}\|u\|_{k-l+\alpha,l}.$$

One can see that $\mathcal{C}^{k+\alpha,j}(\overline{\Omega},P)$ is complete. By assumptions on Ω, P, we see that

$$\mathcal{C}^{k+\alpha,k}(\overline{\Omega},P) \supset \mathcal{C}^{k+\alpha}(\overline{\Omega}\times P).$$

In particular, if $f\in\mathcal{C}^{k+\alpha,j}\cap\mathcal{C}^1$ and $u\in\mathcal{C}^{k+\alpha,j}(\overline{\Omega},P)$, then $f\circ u\in\mathcal{C}^{k+\alpha,j}(\overline{\Omega},P)$ whenever the composition is well-defined. In general, let $\varphi(x,t) = (\tilde\varphi(x,t),t)$ with $\tilde\varphi$ being a map from $\Omega\times P$ into Ω' of class $\mathcal{C}^{k+\alpha,j}\cap\mathcal{C}^1$. Then

$$|v\circ\varphi|_{k+\alpha,j} \leq C(1+|\tilde\varphi|_{1,0}+|\tilde\varphi|_{k+\alpha,j})^{1+k+j}|v|_{k+\alpha,j}.$$

Let $\mathbb{D}$ be the unit disc in $\mathbf{C}$, $\mathbb{D}_r$ the disc of radius r, and $\mathbb{D}_r^+ = \mathbb{D}_r\cap\{\operatorname{Im}z>0\}$. The following result gives coordinate maps in J-holomorphic curves.

Proposition 3.4. *Let $0 < \alpha < 1$ and let $j \geq 1$ be an integer. Let J be an almost complex structure defined by vector fields*

$$X_i = \sum_{1\leq k\leq n} b_{ik}\partial_{\overline{z}_k} + \sum_{1\leq k\leq n} a_{ik}\partial_{z_k}, \quad j = 1, \dots, n.$$

Assume that $A = (a_{ik}), B = (b_{ik})$ are of class $\mathcal{C}^{j+1+\alpha}(\Omega)$. Let $M \subset \Omega$ be a real hypersurface of class $\mathcal{C}^{j+2+\alpha}$. Let $e \colon M \to \mathbf{C}^n$ be a $\mathcal{C}^j$ map such that $e \cdot X = e_1X_1 + \cdots + e_nX_n$ is not tangent to M at each point of M. Let $0 \in M$. There exist two $\mathcal{C}^j$ diffeomorphisms u, R from $\mathbb{D}_r^n$ into Ω satisfying the following.

(i) *For each $t \in \mathbb{D}_r^{n-1}$, $u(\cdot, t)$ is J-holomorphic and embeds $\mathbb{D}_r$ onto $D(t)$.*
(ii) *$u(0, t)$ is in M, and $D(t)$ intersects M transversally along a curve $\gamma(t)$. Also, $u(0) = 0$ and $du(0, t)(\partial_{\overline{\zeta}}) = (e \cdot X)(u(0, t))$.*
(iii) *$R(\cdot, t)$ sends $\mathbb{D}_r^+, (-r, r), \mathbb{D}_r$ into $\Omega^+ \cap D(t), M \cap D(t), D(t)$, respectively. And $R(0) = 0$.*
(iv) *If $A \in \mathcal{C}^{k+\alpha}$, $M \in \mathcal{C}^{k+1+\alpha}$ and $k > j$, the u, R are in $\mathcal{C}^{k+1+\alpha,j}(\mathbb{D}_r, \mathbb{D}_r^{n-1})$.*

Here r depends only on $\inf_\Omega \left|\begin{smallmatrix} B & A \\ \overline{A} & \overline{B}\end{smallmatrix}\right|$, M, e, j, α, $|(A, B)|_{j+\alpha}$, and the diameter of Ω.

Proof. Introducing the new coordinates w by $z = \overline{B}^t(0)w + A^t(0)\overline{w}$, we may assume that $A(0) = 0$ and $B = \mathrm{I}$. Thus we obtain

$$X_j = \partial_{\overline{z}_j} + \sum_{1\leq k\leq n} a_{jk}(z)\partial_{z_k}, \quad j = 1, \dots, n.$$

Applying a unitary change of coordinates, we may assume that T_0M is given by $y_n = 0$. By a change of coordinates which is tangent to the identity and of class $\mathcal{C}^{k+1+\alpha}$, we may assume that M is in $y_n = 0$. By dilation, $\Omega = \mathbb{D}_2^n$ and on it we have $\|A(z)\| < 1/4$ and $|A|_{j+1+\alpha} < 1/C_*$. Here C_* will be determined. Finally, by a dilation in $\mathbb{D}$, we achieve $\|e(x)\| < 1/4$ on M.

Existence in $\mathcal{C}^{j+1+\alpha,j}$ class. We first find u. Recall that $u \colon \mathbb{D} \to \Omega$ is J-holomorphic, if $du(\partial_{\overline{\zeta}})$ is in the span of X_i's. Then the equations are

$$\partial_{\overline{\zeta}} u_i = \sum_{1\leq l\leq n} a_{li}(u)\overline{\partial_\zeta u_l}, \quad i = 1, \dots, n.$$

In column vectors, they become

$$\partial_{\overline{\zeta}} u = A^t(u)\overline{\partial_\zeta u}. \tag{3.6}$$

At the origin, $X_j(0) = \partial_{\overline{z}^j}$. Let $\tilde{e}_1, \dots, \tilde{e}_{n-1}$ be the standard base of $\mathbf{C}^{n-1} \times 0$. Then $\tilde{e}_1, \dots, \tilde{e}_{n-1}$, $e(0, t)$ are $\mathbf{C}$-linearly independent. For $t \in P = \mathbb{D}^{n-1}$, we look for a J-holomorphic curve u satisfying $u = \Psi(u)$ with

$$\Psi(u)(\zeta) = t \cdot (\tilde{e}_1, \dots, \tilde{e}_{n-1})/n + \zeta\overline{e(0, t)} + \Phi(u) - P_1\Phi(u). \tag{3.7}$$

Here $\Phi(u) = T_{\mathbb{D}}(A^t(u)\overline{\partial_\zeta u})$ and $P_1\Phi(u)(\zeta, t) = \Phi(u)(0, t) + \partial_\zeta\Phi(u)(0, t)\zeta$. Let $\mathcal{B}_1$ be the closed unit ball in $\mathcal{B} = [\mathcal{C}^{j+1+\alpha,j}(\overline{\mathbb{D}}, P)]^n$ equipped with norm $|u|_{j+1+\alpha,j}$. When $u \in \mathcal{B}_1$, $\Phi(u)$ is in $\mathcal{B}$ (see Lemma 2.1). Then $P_1\Phi(u)$ is continuous and of class $\mathcal{C}^j$ in t. It is also a polynomial in ζ. In particular, $P_1\Phi(u)$ and $\Psi(u)$ are in $\mathcal{B}$. One can verify that if

$|A|_{j+1+\alpha} < 1/C_*$ on $\mathbb{D}_2 \times P$, then $u \to \Psi(u)$ is a contraction map from $\mathcal{B}_1$ into itself (here we need A to be in $\mathcal{C}^{j+1+\alpha}$ instead of $\mathcal{C}^{j+\alpha}$). We take $u \in \mathcal{B}_1$ to be its fixed point.

Recall that after dilation, $|A|_{j+1+\alpha} < 1/C_*$ and $j \geq 1$. Then $\Phi(u)$ and $P_1(\Phi(u))$ have small $\mathcal{C}^1$ norms on $\mathcal{B}_1$ in ζ, t such that u is a $\mathcal{C}^1$ diffeomorphism in ζ, t.

Higher order derivatives. We have obtained a solution $u \in \mathcal{C}^{j+1+\alpha,j}(\overline{\mathbb{D}}, P)$. Assume now that $A \in \mathcal{C}^{k+\alpha}(\mathbb{D}_2^n)$ with $k > j$. We want to show a stronger result: Assume that for all $l \leq j$, $\partial_t^l u(\zeta, t)$ are continuous on $\mathbb{D} \times P$ and distributional derivatives $\partial_\zeta \partial_t^l u(\cdot, t)$ have bounded $L^p(\mathbb{D})$ norms on P with $p > 2$. Assume that $u(\cdot, t)$ is J-holomorphic on $\mathbb{D}$. Then $u \in \mathcal{C}^{k+1+\beta,j}(\mathbb{D}_r, P)$ for $r < 1$ and $\beta = \min(\alpha, 1 - 2/p)$.

Indeed, (3.6) implies that the first-order derivatives of $\partial_t^l u(\cdot, t)$ have bounded $L^p(\mathbb{D})$ norms on P. By Morrey's inequalities, $u \in \hat{\mathcal{C}}^{\beta,j}(\mathbb{D}_r, P)$ for any $r < 1$. (See Lemma 7.16 and Theorem 7.17 in [4], pp. 162-163.)

Fix $\zeta_0 \in \mathbb{D}$. Let $u = \tilde{u} + A^t(u(\zeta_0, t))\overline{\tilde{u}}$ and $\tilde{u}(\zeta_*, t) = u_*(\zeta, t)$ with $\zeta_* = \zeta_0 + \mu\zeta$. Here $0 < \mu < \frac{1}{2}(1 - |\zeta_0|)$ will be determined. We get on $\mathbb{D}$

$$\partial_{\overline{\zeta}} u_* = A_*^t(\zeta, t)\overline{\partial_\zeta u_*}, \quad A_*(0, t) = 0, \tag{3.8}$$

$$A_*(\zeta, t) = [A(u(\zeta_*, t)) - A(u(\zeta_0, t))][\mathrm{I} - \overline{A(u(\zeta_0, t))}A(u(\zeta_*, t))]^{-1}. \tag{3.9}$$

Let χ be a smooth function with support in $\mathbb{D}_{1/4}$. Let $v = \chi u_*$. Multiply (3.8) by χ and rewrite it as

$$\partial_{\overline{\zeta}} v - A_*^t(\zeta, t)\overline{\partial_\zeta v} = u_* \partial_{\overline{\zeta}} \chi - A_*^t(\zeta, t)\overline{u_* \partial_\zeta \chi}. \tag{3.10}$$

Let $\tilde{\chi}$ be a smooth function with compact support in $\mathbb{D}$. We also assume that $\tilde{\chi} = 1$ on $\mathbb{D}_{1/4}$ and $|\tilde{\chi}|_1 < 5$. Replacing A_* by $\tilde{\chi}A_*$, we may assume that $A_*(\cdot, t)$ has compact support in $\mathbb{D}$. Using (3.9), we get for $\zeta, \zeta' \in \mathbb{D}$,

$$|A_*(\zeta, t)| \leq C|A(u(\cdot, t))|_\beta \mu^\beta,$$
$$|A_*(\zeta', t) - A_*(\zeta, t)| \leq C|A(u(\cdot, t))|_\beta \mu^\beta |\zeta' - \zeta|^\beta.$$

Therefore, $\|A_*\|_{\beta,0} \leq C|A \circ u|_{\beta,0}\mu^\beta < \epsilon_\beta$. Here ϵ_β is the constant in Lemma 2.2 and μ is sufficiently small. Apply $T = T_{\mathbb{D}}$ to (3.10). Since v has compact support, then

$$v - T(A_*^t \overline{\partial_\zeta v}) = T(u_* \partial_{\overline{\zeta}} \chi - A_*^t \overline{u_* \partial_\zeta \chi}).$$

Write the right-hand side as w and solve for $v = (\mathrm{I} - TA_*^t \overline{\partial_\zeta})^{-1} w$. Since u_* is in $\hat{\mathcal{C}}^{\beta,j}(\overline{\mathbb{D}}, P)$, then A_*, w are in $\hat{\mathcal{C}}^{\beta,j}(\overline{\mathbb{D}}, P)$. By Lemma 2.2, v and hence u are in $\hat{\mathcal{C}}^{1+\beta,j}$. Repeating the procedure, we get $u \in \hat{\mathcal{C}}^{k+1-j+\beta,j}$. Also $u \in \hat{\mathcal{C}}^{k+1-l+\beta,l}$ for all $l \leq j$. This shows that $u \in \mathcal{C}^{k+1+\beta,j}$.

End of the proof. We assume that $\Omega = \mathbb{D}_2^n$ and that Ω^+, M are subsets defined by $y_n > 0$ and $y_n = 0$, respectively. Let $\overline{e(0, t)} = (a, b' + ib'')$. Since $e(0, t) \cdot \partial_{\overline{\zeta}}$ is not tangent to M, then $b' + ib'' \neq 0$. Without loss of generality, we may assume that $b' \geq |b''|$. We have $u = \Psi(u)$. By (3.7), $D(t) \cap M$ is defined by

$$b''\xi + b'\eta = F(\xi, \eta, t), \quad F(\xi, \eta, t) = \mathrm{Im}\{P_1\Phi_n(u(\xi + i\eta, t)) - \Phi_n(u(\xi + i\eta, t))\}. \tag{3.11}$$

We already know that $F \in \mathcal{C}^{k+1+\alpha,j}(\mathbb{D}_r, P)$. We may also achieve $|\partial_\eta F| < b'/2$, by assuming $|A|_{j+\alpha} < 1/C_*$. By the implicit function theorem, (3.11) has a solution $\eta = h(\xi, t)$ for

$|\xi| < r/c, t \in P$. Now

$$(\partial_\xi h, \partial_t h) = (b' - \partial_\eta F(\xi,\eta,t))^{-1}\big(\partial_\xi F - b'', \partial_t F\big)$$

implies that $\partial_t^l h \in \mathcal{C}^{k+1+\alpha-l}$ for all $l \leq j$. On $\mathbb{D}_{r/c} \times P$, define

$$R(\zeta,t) = u(\xi + i(\eta + h(\xi,t)), t).$$

Then $R(\cdot,t)$ sends $\mathbb{D}^+_{r/c}$ into $D^+(t)$. Replace $R(\zeta,t)$ by $R(\zeta/c,t)$. The remaining assertions can be verified easily. □

We remark that the above R is not J-holomorphic.

It is well-known that via the Fourier transform, the boundedness of derivatives of a function on all lines parallel to coordinates axes yields some smoothness of the function in all variables (see Rudin [11], p. 203). To limit the loss of derivatives, we will use the Fourier transform only on curves. This requires us to bound derivatives of a function on a larger family of curves.

Let γ be a $\mathcal{C}^k$ curve in $\mathbf{R}^n$, and let f be a function of class $\mathcal{C}^k$ on $\mathbf{R}^n$. We have

$$\partial_t^k f(\gamma(t)) = (\gamma'(t)\cdot\partial)^k f)(\gamma(t)) + \sum_{1\leq|\beta|<k} Q_{k,\beta}(\partial_t^{(k+1-|\beta|)}\gamma)(\partial^\beta f)(\gamma(t)). \tag{3.12}$$

Here $Q_{k,\beta}$ are polynomials, $\partial^{(k)}$ denotes derivatives of order $\leq k$, and

$$v\cdot\partial = v_1\partial_{x_1} + \cdots + v_n\partial_{x_n}.$$

Lemma 3.5. *Let k be a positive integer and let $\epsilon > 0$.*

(i) There exist N vectors $v_j = (1, v_j') \in \mathbf{R}^n$ such that $|v_j'| < \epsilon$ and

$$\partial^\alpha = c_{\alpha,1}(v_1\cdot\partial)^k + \cdots + c_{\alpha,N}(v_N\cdot\partial)^k, \quad |\alpha| = k. \tag{3.13}$$

(ii) If $v_1,\ldots,v_N$ satisfy (3.13), there exists $\delta > 0$ such that if $|u - v| < \delta$, then

$$\partial^\alpha = Q_{\alpha,1}(u)(u_1\cdot\partial)^k + \cdots + Q_{\alpha,N}(u)(u_N\cdot\partial)^k, \quad |\alpha| = k. \tag{3.14}$$

Here $Q_{\alpha,j}$ are rational functions with $Q_{\alpha,j}(v) = c_{\alpha,j}$. And N depends only on k, n.

Proof. (i). Equivalently, we need to verify (3.13)-(3.14) when ∂ is replaced by $\xi \in \mathbf{R}^n$. It holds for $n = 1$. Assume that it holds when n is replaced by $n-1$. For ξ_n^k, we take distinct non-zero constants $\lambda_1,\ldots,\lambda_k$. Then ξ_n^k is in the linear span of ξ_1^k, $(\xi_1 + \lambda_1\xi_n)^k, \ldots, (\xi_1 + \lambda_k\xi_n)^k$. Let $\xi_n^j P(\xi_1,\ldots,\xi_{n-1})$ be a monomial of degree $k > j$. Then by the induction assumption

$$\xi_n^j P(\xi_1,\ldots,\xi_{n-1}) = \xi_n^j[c_1(v_1\cdot\xi)^{k-j} + \cdots + c_l(v_l\cdot\xi)^{k-j}].$$

Here $v_j = (1, v_j'', 0)$ with $|v_j''| < \epsilon/2$. Then $\xi_n^i(v_l\cdot\xi)^{k-i}$ are in the linear span of $(v_l\cdot\xi)^k$, $(v_l\cdot\xi + \lambda_j\xi_n)^k$ with $j = 1,\ldots,k$. Note that λ_j can be arbitrarily small. Thus, (i) is verified.

(ii). For $|\alpha| = k$ we have expansions

$$\xi^\alpha = \sum_{1\leq j\leq N} c_{\alpha,j}(v_j\cdot\xi)^k, \quad \xi^\alpha = \sum_{1\leq j\leq N} c_{\alpha,j}(u_j\cdot\xi)^k + \sum_{|\beta|=k}\widetilde{Q}_{\alpha\beta}(v-u)\xi^\beta.$$

Clearly, $\widetilde{Q}_{\alpha\beta}(0) = 0$. Moving the last sum to the left-hand side and inverting $\mathrm{I} - (\widetilde{Q}_{\alpha\beta})$ yields (3.14). □

We now use (3.12) to estimate partial derivatives via derivatives on curves. Set $t' = (t_2, \dots, t_n)$ and $t = (t_1, t')$.

Proposition 3.6. *Let k, N be positive integers. For $1 \leq j \leq N$, let R_j be $\mathcal{C}^1$ diffeomorphisms from $\Omega_j \subset \mathbf{R}^n$ onto an open subset Ω of $\mathbf{R}^n$. Assume that $R_j(\cdot, t') \in \mathcal{C}^k$ and $R_j(0) = 0$. Suppose that at $0 \in \Omega$*

$$\partial^\alpha = \sum_{1 \leq j \leq N} c_{\alpha,j} (\partial_{t_1} R_j(0) \cdot \partial)^{|\alpha|}, \quad 1 \leq |\alpha| \leq k. \tag{3.15}$$

Let $f \in \mathcal{C}^0(\Omega)$. Then the following hold.

(i) Let f be of class $\mathcal{C}^k$ near $0 \in \Omega$. There exist rational functions $Q_{\alpha,i,j}$ such that for $x = R_j(t^j)$ near 0 and $|\alpha| = m \leq k$ with $m \geq 1$,

$$\partial^\alpha f(x) = \sum_{i=1}^{m} \sum_{j=1}^{N} Q_{\alpha,i,j} \Big(\partial_{t_1^1}^{(m-i+1)} R(t^1), \dots, \partial_{t_1^N}^{(m-i+1)} R(t^N) \Big) \partial_{t_1^j}^i f(R_j(t^j)). \tag{3.16}$$

(ii) Suppose that R_j are affine, i.e. $R_j(t) - R_j(y) = R_j(t-y)$ wherever they are defined. Suppose that $L^\infty_{t_1}$ norms of one-dimensional distributions $\partial^m_{t_1}(f \circ R_j)(\cdot, t')$ are bounded in t' for all $m \leq k$. Then near 0, $\partial^\alpha f$ are Lipschitz functions for all $|\alpha| < k$.

(iii) Let R_j be of class $\mathcal{C}^{k+1}$ near $0 \in \mathbf{R}^n$ and let $n < p < \infty$. Suppose that $L^p_{t_1}$ norms of one-dimensional distributions $\partial^m_{t_1}(f \circ R_j)(\cdot, t')$ are bounded in t' for all $m \leq k$. Then near 0, f is of class $\mathcal{C}^{k - \frac{n}{p}}$.

Proof. (i) follows from (3.12) and (3.14), by hypothesis (3.15).

(ii). Applying dilation and replacing f by χf, we may assume that f has compact support in Δ^n. Let $\chi_\epsilon(x) = \epsilon^{-n}\chi(\epsilon^{-1}x)$ for a smooth function χ with support in Δ^n and $\int \chi \, dx = 1$. Let $f_\epsilon(x) = \int f(y)\chi_\epsilon(x-y)\, dy$ and $f_{\epsilon,j} = f_\epsilon \circ R_j$.

Changing variables via R_j, we get

$$f_{\epsilon,j}(t) = \int f(R_j(t) - R_j(y))\chi_\epsilon(R_j(y)) \det R_j'(y)\, dy.$$

Using $R_j(t) - R_j(y) = R_j(t-y)$, we get $|f_{\epsilon,j}(\cdot, t')|_k < C$ for C independent of ϵ and t'. In (3.16), we substitute f_ϵ for f. Therefore, $\partial^\alpha f_\epsilon$ are bounded near 0. We can find a sequence f_{ϵ_j} such that as ϵ_j tends to 0, $\partial^\alpha f_{\epsilon_j}$ converges uniformly for $|\alpha| < k$, and the Lipschitz norms of $\partial^\alpha f_{\epsilon_j}$ are bounded by a constant. Since f_ϵ converges to f uniformly as $\epsilon \to 0^+$ then $\partial^{k-1} f \in Lip_{loc}$.

(iii). For the f, we define a distribution $T_j f$ by

$$T_j f(\phi) = (-1)^k \int_{\mathbf{R}^n} f \circ R_j(t) \partial_{t_1}^k (\phi(R_j(t)))\, dt.$$

Here ϕ are test functions supported in Δ^n_ϵ with ϵ small. It is clear that defined near 0, $T_j f$ is a distribution of order $(\leq)k$. Integrating in t_1-variable first and throwing the

one-dimensional derivative onto $f\circ R_j$ yields

$$|T_jf(\phi)| \leq C\int_{\mathbf{R}^{n-1}} \|\partial_{t_1}^k[f\circ R_j](\cdot,t')\|_{L^p_{t_1}}\|\phi\circ R_j(\cdot,t')\|_{L^q_{t_1}}\,dt'$$
$$\leq C_1\int \|\phi\circ R_j(\cdot,t')\|_{L^q_{t_1}}\,dt' \leq C_2\|\phi\circ R_j\|_{L^q}\leq C_3\|\phi\|_{L^q}.$$

Here the second last inequality is obtained from the Hölder inequality and $\operatorname{supp}\phi\subset\Delta^n_\epsilon$. Hence near 0, $T_jf\in L^p$ when $p>1$. Next we find a differential operator $P_{j,k}(\partial)$ of order k such that $P_{j,k}(\partial)f=T_jf$. To find it, we use a smooth function g to obtain

$$T_jg(\phi)=\int\partial_{t_1}^k[g\circ R_j(t)]\phi(R_j(t))\,dt=\int(\phi\tilde P_{j,k}(\partial)g)\circ R_j(t)\,dt$$
$$=\int[\det((R_j^{-1})')\tilde P_{j,k}(\partial)g]\phi\,dx\overset{\text{def}}{=\!=}(P_{j,k}(\partial)g)(\phi).$$

Since $R_j\in\mathcal{C}^{k+1}$, it is easy to see that

$$P_{j,k}(\partial)=\det((R_j^{-1})')\tilde P_{j,k}(\partial)=\sum_{|\alpha|\leq k}a_{j,k,\alpha}\partial^\alpha,\quad a_{j,k,\alpha}\in\mathcal{C}^{|\alpha|}.$$

The last assertion implies that $P_{j,k}(\partial)$ has order k. The definition of T_jf and identity $P_{j,k}(\partial)g=T_jg$ implies that as distributions defined near 0, $P_{j,k}(\partial)f=T_jf$.

Note that

$$\sum_{|\alpha|=k}a_{j,k,\alpha}(x)\partial_x^\alpha=\sum_{|\alpha|=k}C_\alpha\det((R_j^{-1})')(\partial_{t_1^j}R_j(t^j))^\alpha\partial_x^\alpha,\quad C_\alpha\neq0.$$

Here $t^j=R_j^{-1}(x)$. Combining with (3.15), we get for $g\in\mathcal{C}^k$ and $1\leq|\alpha|\leq k$,

$$\partial^\alpha g=\sum_{1\leq i\leq m}\sum_{1\leq j\leq N}b_{\alpha,j,i}P_{j,i}(\partial)g,\quad b_{\alpha,j,i}\in\mathcal{C}^i.$$

The last assertion, combined with $\operatorname{ord}P_{j,i}(\partial)\leq i$, $a_{j,k,\alpha}\in\mathcal{C}^{|\alpha|}$ and $P_{j,i}(\partial)f\in L^p$, implies that near 0, $\partial^\alpha f$ are in L^p for $1\leq|\alpha|\leq k$; by a Sobolev embedding theorem ([7], p. 123), $f\in\mathcal{C}^{k-1+\beta}$ with $\beta=1-\frac{n}{p}$. □

4. Cauchy-Green operator on domains with parameter

The following result is certainly classical; see [13], section 8.1 (pp. 56-61). For the convenience of the reader, we present details for a parameter version. Recall that P is the closure of a bounded open set in a euclidean space and two points a,b in P can be connected by a smooth curve in P of length at most $C|b-a|$.

Lemma 4.1. *Let τ be a complex-valued function on $\overline{\mathbb{D}}^+\times P$ of class $\mathcal{C}^{k+1+\alpha,0}(\overline{\mathbb{D}}^+,P)$. Suppose that for $z,z'\in\mathbb{D}^+$ and $t\in P$,*

$$|\tau(z',t)-\tau(z,t)|\geq|z'-z|/C. \tag{4.1}$$

(*i*) *Let f be a continuous function on $[-1,1]\times P$. Let*

$$C_0f(z,t)=\frac{1}{2\pi i}\int_{-1}^1\frac{f(s,t)}{\tau(s,t)-\tau(z,t)}\,ds,\quad z\in\mathbb{D}^+.$$

Then $|\partial_z^k C_0 f(z,t)| \leq C_k |f|_0 / |\operatorname{Im} z|^{k+1}$, *where* c_k *depends only on* $\|\tau\|_{k,0}$.

(*ii*) *If* f *is a function of class* $\mathcal{C}^{k+\alpha,0}([-1,1],P)$, *then* $C_0 f$ *extends continuously to* $(\mathbb{D} \cup (-1,1)) \times P$. *Moreover,* $C_0 f \in \mathcal{C}^{k+\alpha,0}(\overline{\mathbb{D}}_r^+, P)$ *for* $r < 1$ *with* $|C_0 f|_{k+\alpha,0} \leq C|f|_{\alpha,0}$.

(*iii*) *Let* f *be a function of class* $\mathcal{C}^{k+\alpha,0}(\overline{\mathbb{D}}^+, P)$. *For* $z \in \mathbb{D}^+$, *define*

$$S_0 f(z,t) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \int_{\{\zeta \in \mathbb{D}^+ : \, |\tau(\zeta,t) - \tau(z,t)| > \epsilon\}} \frac{f(\zeta,t)}{(\tau(\zeta,t) - \tau(z,t))^2} \, d\xi d\eta,$$
$$T_0 f(z,t) = -\frac{1}{\pi} \int_{\mathbb{D}^+} \frac{f(\zeta,t)}{\tau(\zeta,t) - \tau(z,t)} \, d\xi d\eta.$$

Then $S_0 f \in \mathcal{C}^{k+\alpha,0}(\overline{\mathbb{D}}_r^+, P)$ *and* $T_0 f \in \mathcal{C}^{k+1+\alpha,0}(\overline{\mathbb{D}}_r^+, P)$ *for* $r < 1$ *with* $|S_0 f|_{k+\alpha,0} + |T_0 f|_{k+1+\alpha,0} \leq C|f|_{k+\alpha,0}$.

Proof. (i). Note that (4.1) implies that $|\tau(z,t) - \tau(s,t)| \geq \operatorname{Im} z / C$ for $-1 \leq s \leq 1$ and $z \in \mathbb{D}^+$. The proof is straightforward by taking derivatives in $z, \overline{z}$ directly onto the kernel.

(ii). Let $z = x + iy$. Let χ be a smooth function with compact support in $(-1,1)$. Replacing $f(x,t)$ with $\chi(x) f(x,t) / \partial_x \tau(x,t)$, it suffices to get the norm estimate on $\mathbb{D}_r \times P$ for

$$C_0 f(z,t) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}^+} \frac{f(\zeta,t)}{\tau(\zeta,t) - \tau(z,t)} \, d\tau(\zeta,t) \tag{4.2}$$
$$= \frac{1}{2\pi i} \int_{\partial \mathbb{D}^+} \frac{f(\zeta,t) - f(x,t)}{\tau(\zeta,t) - \tau(z,t)} \, d\tau(\zeta,t) + \epsilon f(x,t).$$

Here the differentiation and integration are in ζ. And $\epsilon = 1$, if $\tau(\cdot,t)$ preserves the orientation of $\mathbb{D}^+$; otherwise $\epsilon = -1$. From (4.1) and $\tau \in \mathcal{C}^{1,0}(\mathbb{D}^+, P)$, we know that ϵ is independent of t. Let $C_1 f$ denote the second integral in (4.2). We denote a j-th derivative in x, y by ∂^j. In what follows, the norms $|\cdot|_{j+\alpha,0}$ for f, τ are on $\mathbb{D}^+$, and norms $|\cdot|_{j+\alpha,0}$ for $C_0 f$ are on $\mathbb{D}_r^+$ with $r < 1$. These norms will be denoted by the same notation $|\cdot|_{j+\alpha}$. Since t is fixed, we suppress it in all expressions. All constants are independent of t.

That $C_1 f$ extends continuously to $\overline{\mathbb{D}}^+ \times P$ follows from the continuity of f and

$$\left| \frac{f(s) - f(x)}{\tau(s) - \tau(z)} \right| \leq C|f|_\alpha |x - s|^{\alpha - 1}.$$

Take (4.2) as the definition of C_0. Differentiating it gives

$$\partial C_0 f(z) = \frac{\partial \tau(z)}{2\pi i} \int_{\partial \mathbb{D}^+} \frac{f(\zeta) - f(x)}{(\tau(\zeta) - \tau(z))^2} \, d\tau(\zeta). \tag{4.3}$$

Using $|\tau(s) - \tau(z)| \geq (|s - x| + |y|)/C$, we get

$$|\partial C_0 f(z)| \leq \|\tau\|_1 \int_{\partial \mathbb{D}^+} \frac{C|f|_\alpha |s - x|^\alpha}{|y|^2 + |s - x|^2} \, ds \leq C'_\alpha |f|_\alpha |y|^{\alpha - 1}.$$

By a type of Hardy-Littlewood lemma, we obtain $|C_0 f|_{\alpha,0} \leq C|f|_{\alpha,0}$. For higher derivatives of $C_0 f$, we differentiate (4.2) in z variable and transport derivatives to f via integration

by parts. We get for $|I|=k$

$$\partial^I C_0 f(z)=\sum_{1\leq |J|\leq |I|}\frac{\partial^J\tau(z)}{2\pi i}\int_{\partial\mathbb{D}^+}\frac{f_{IJ}(\zeta)}{\tau(\zeta)-\tau(z)}\,d\tau(\zeta). \tag{4.4}$$

Here $f_{IJ}(s)$ are polynomials in $(\partial_s\tau(s))^{-1},\partial_s^l f(s),\partial_s^{l+1}\tau(s)$ with $l\leq k$. As before, we have the continuity of

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+}\frac{f_{IJ}(\zeta)}{\tau(\zeta)-\tau(z)}\,d\tau(\zeta)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+}\frac{f_{IJ}(\zeta)-f_{IJ}(x)}{\tau(\zeta)-\tau(z)}\,d\tau(\zeta)+\epsilon f_{IJ}(x).$$

Differentiating the integral in (4.4) one more time we get a formula analogous to (4.3). As in case $k=0$, we can verify that the $\mathcal{C}^\alpha$ norms of $\partial^k C_0 f(\cdot,t)$ on $\overline{\mathbb{D}}_r$ are bounded.

(iii). We first show that $S_0 f\in\mathcal{C}^{\alpha,0}(\overline{\mathbb{D}}^+,P)$.

Let $-2id\xi\wedge d\eta=A(\zeta,t)\,d\tau(\zeta,t)\wedge d\overline{\tau(\zeta,t)}$. Let χ be a smooth function with compact support in $\mathbb{D}^+_{r'}\cup(-r',r')$, $0<r<r'<1$. Replace $f(\zeta,t)$ by $\chi(\zeta)f(\zeta,t)A(\zeta,t)$. We may reduce to the case that $f(\cdot,t)$ is supported in $\overline{\mathbb{D}}^+_{r'}$ with $r<r'<1$. We may also replace the domain of integration by a smooth domain D with $\mathbb{D}^+_r\subset\overline{D}\subset\overline{\mathbb{D}}^+_{r'}$. Again, we suppress the parameter t in all expressions and write

$$S_0 f(z)=\frac{1}{2\pi i}\int_D\frac{(f(\zeta)-f(z))\,d\tau(\zeta)\wedge d\overline{\tau(\zeta)}}{(\tau(\zeta)-\tau(z))^2}-\frac{f(z)}{2\pi i}\int_{\partial D}\frac{d\overline{\tau(\zeta)}}{\tau(\zeta)-\tau(z)}. \tag{4.5}$$

On ∂D, write $d\overline{\tau(\zeta)}=a_0(\zeta)\,d\tau(\zeta)$. By (ii), we know that the last integral in (4.5) is in $\mathcal{C}^{\alpha,0}(\overline{\mathbb{D}}^+_r,P)$. Name the first integral in (4.5) by $\tilde g(z)/(2\pi i)$. That $\tilde g$ extends continuously follows from the continuity of f and $|f(\zeta)-f(z)|/|\tau(\zeta)-\tau(z)|^2\leq C|\zeta-z|^{\alpha-2}$. Write

$$\begin{aligned}\tilde g(z_2)-\tilde g(z_1)&=\int_D\frac{(f(z_1)-f(z_2))\,d\tau(\zeta)\wedge d\overline{\tau(\zeta)}}{(\tau(\zeta)-\tau(z_2))(\tau(\zeta)-\tau(z_1))}\\&\quad+\int_D\frac{(f(\zeta)-f(z_2))(\tau(z_2)-\tau(z_1))}{(\tau(\zeta)-\tau(z_2))^2(\tau(\zeta)-\tau(z_1))}\,d\tau(\zeta)\wedge d\overline{\tau(\zeta)}\\&\quad+\int_D\frac{(f(\zeta)-f(z_1))(\tau(z_2)-\tau(z_1))}{(\tau(\zeta)-\tau(z_1))^2(\tau(\zeta)-\tau(z_2))}\,d\tau(\zeta)\wedge d\overline{\tau(\zeta)}.\end{aligned}$$

The last two integrals can be estimated by a standard argument for Hölder estimates, bounded in absolute value by $C_\alpha\|f\|_{\alpha,0}|z_2-z_1|^\alpha$. The first integral can be rewritten as the product of $f(z_1)-f(z_2)$ and $\mathcal{I}$ for

$$\begin{aligned}\mathcal{I}&=\frac{1}{\tau(z_2)-\tau(z_1)}\int_D\left\{\frac{d\tau(\zeta)\wedge d\overline{\tau(\zeta)}}{\tau(\zeta)-\tau(z_2)}-\frac{d\tau(\zeta)\wedge d\overline{\tau(\zeta)}}{\tau(\zeta)-\tau(z_1)}\right\}\\&=2\pi i\frac{\overline{\tau(z_2)-\tau(z_1)}}{\tau(z_1)-\tau(z_2)}+\frac{1}{\tau(z_1)-\tau(z_2)}\int_{\partial D}\left\{\frac{\overline{\tau(\zeta)}\,d\tau(\zeta)}{\tau(\zeta)-\tau(z_2)}-\frac{\overline{\tau(\zeta)}\,d\tau(\zeta)}{\tau(\zeta)-\tau(z_1)}\right\}.\end{aligned}$$

A derivative of $\int_{\partial D}\frac{\overline{\tau(\zeta)}\,d\tau(\zeta)}{\tau(\zeta)-\tau(z)}$ is $\partial_z\tau(z)\int_{\partial D}\frac{d\overline{\tau(\zeta)}}{\tau(\zeta)-\tau(z)}$ which, by (ii), is bounded. By the mean-value-theorem, the last term in $\mathcal{I}$ is bounded. This shows that $S_0 f\in\mathcal{C}^{\alpha,0}(\overline{\mathbb{D}}^+_r,P)$.

For higher order derivatives, we transport derivatives to f. Define $h_*(\tau(z)) = h(z)$ and $\omega(t) = \tau(\cdot, t)(D)$. Let $C_* = C_{\partial\omega(t)}$, $T_* = T_{\omega(t)}$ and $S_* = S_{\omega(t)}$. Rewrite (4.5) as $g_*(\tau) = S_* f_*$. Integrating by parts, we obtain

$$g_*(\tau) = \frac{1}{2\pi i} \int_{\omega(t)} \frac{\partial_\varsigma f_*(\varsigma)}{\varsigma - \tau} \, d\varsigma \wedge d\overline{\varsigma} - \frac{1}{2\pi i} \int_{\partial\omega(t)} \frac{f_*(\varsigma)}{\varsigma - \tau} \, d\overline{\varsigma}.$$

On $\partial\omega(t)$, we write $d\overline{\tau} = a(\tau, t)\, d\tau$ with $a \in \mathcal{C}^{k+\alpha,0}(\partial D, P)$. Taking derivatives, we get

$$\partial_{\overline{\tau}} S_* f = \partial_\tau f_*, \quad \partial_\tau S_* f_* = S_* \partial_\tau f_* - \partial_\tau C_* a f_*.$$

Using the last formula k times, we get

$$(\partial_\tau)^k S_* f_* = S_* \partial_\tau^k f_* - \sum_{0 \le j < k} \partial_\tau^{k-j} C_* a \partial_\tau^j f_*.$$

We return to the z coordinates. Let $\tilde{a}(z,t) = a(\tau(z,t), t)$. Let ∂_z^K be a derivative in $z, \overline{z}$ of order k. Let $\partial^{(j)}$ denote derivatives of orders $\le j$. Then

$$\begin{aligned} \partial_z^K S_0 f(z) &= p^1\big(\partial_z^{(k)}\tau\big) \cdot \big(\partial_\tau^{(k)} S_* f_*, \partial_\tau^{(k)} f_*\big) \circ \tau \\ &= \sum_{0 \le j \le k} p_j^2\big(\partial_z^{(k)}\tau\big) \cdot \big(S_* \partial_\tau^{(k)} f_*, \partial_\tau^{(k-j)} C_*(a \partial_\tau^{(j)} f_*), \partial_\tau^{(k)} f_*\big) \circ \tau \\ &= \sum_{0 \le j \le k} q_{j,k}^1 \cdot \big(S_0 q_k^2 \partial_z^{(k)} f, \partial_\tau^{(k-j)} C_0(\tilde{a} q_j^3 \partial_\tau^{(j)} f), \partial_\tau^{(k)} f\big). \end{aligned} \tag{4.6}$$

Here integral operator S_0 is over the domain D. And C_0 is over ∂D. p_j^i are vectors of polynomials, and $q_{l,j}^1, q_j^i$ are matrices of polynomials in $(\det \tau')^{-1}, \partial_z^{(j)}\tau$.

That $S_0 f \in \mathcal{C}^{k+\alpha,0}(\overline{\mathbb{D}}^+, P)$ follows from the assertion for $k = 0$ and (ii).

Note that $\partial_\tau T_* = S_*$ and $\partial_{\overline{\tau}} T_* = \mathrm{I}$. Thus, $T_0 f \in \mathcal{C}^{k+1+\alpha,0}(\overline{\mathbb{D}}_r^+, P)$ by (ii), the product rule, and the chain rule as used in (4.6). $\square$

5. Proof of the higher dimensional result

Let $\Delta_r^n, \Delta_r^{2n-1}, \Delta_r^{2n}$ be the polydiscs of radius r in the x-subspace, hyperplane $y_n = 0$, and $\mathbf{R}^{2n}$, respectively.

In this section, we will prove Theorem 1.2. In view of Lemma 3.1 and Example 3.2, it is worth stating a more general result. This will also make the proof transparent.

Theorem 5.1. *Let $k \ge 4$ be an integer. Let $\Omega_1, \Omega_2, M, \alpha$ be as in Theorem 1.2 with $M \in \mathcal{C}^{k+1+\alpha}$. For $i = 1, 2$, let J^i be an almost complex structure of class $\mathcal{C}^{k+\alpha}(\Omega_i \cup M)$ on $\Omega_i \cup M$. Suppose that at each point $p \in M$ there is a tangent vector $v_p \in T_pM$ such that $J_p^1 v_p, J_p^2 v_p$ are in the same connected component of $T_p\mathbf{R}^{2n} \setminus T_pM$. Let $f \in \mathcal{C}^0(\Omega_1 \cup \Omega_2)$ be a continuous function on $\Omega_1 \cup M \cup \Omega_2$ such that $(\partial_{x_j} + \sqrt{-1} J^i \partial_{x_j}) f$ and $(\partial_{y_j} + \sqrt{-1} J^i \partial_{y_j}) f$, defined on Ω_i, extend to functions in $\mathcal{C}^k(\Omega_i \cup M)$ for $i = 1, 2$, $j = 1, \dots, n$. Then f is of class $\mathcal{C}^{k-3+\beta}(\Omega_1 \cup M)$ for all $\beta < 1$.*

Notice that no integrability condition is assumed. A by-product of our proof is $f \in \mathcal{C}_{loc}^{k-3+\beta}(\Omega_1)$ for all $\beta < 1$ when $k \ge 3$. (Of course the assumptions on f, J^2 for M, Ω_2 are not needed.)

The main ingredients of the proof are in the following.

Step 1. We will show that the Fourier transform of f on lines L in M decays in the ξ-variable. To use the differential equations for f, lines L need to be transversal to the complex tangent vectors of M of both structures. Two almost complex structures yield decay of the Fourier transform at opposite rays. This is the only place we need both structures. This gives us smoothness of f on M.

Step 2. We will obtain smoothness of f on each side of M (up to the boundary) via the one-sided almost complex structure. We attach a family of holomorphic discs to M with respect to the structure. Such a disc will have regularity as good as the structure provides. This is achieved by extending the structure to a neighborhood of M. The regularity of f on M yields uniform bounds of pointwise derivatives of f along the discs up to their boundaries in M.

Step 3. Let $\Omega^+ = \Omega_1$. After obtaining smoothness of f on families of discs in $\Omega^+ \cup M$, we conclude the smoothness of f on of $\Omega^+ \cup M$ via Proposition 3.6.

We now carry out details. We need a preparation for Step 1.

Step 0. Match approximate J-holomorphic half-discs in M.

We may assume that M is $\Delta^{2n-1} \times 0$, $\Omega^+ = \Delta^{2n} \cap \{y_n > 0\}$ and $\Omega^- = \Delta^{2n} \cap \{y_n < 0\}$. Let $0 < r < 1$ be sufficiently small. By the assumption, there is a vector $v_0 \in T_0M$ such that the vectors $J_0^1 v_0, J_0^2 v_0$ are transversal to T_0M and are in Ω^+. Thus the line segments $tJ_0^1 v_0, tJ_0^2 v_0$ $(0 < t \leq 1)$ are transversal to M and are in Ω^+, by shrinking v if necessary. Here we have identified $\mathbf{R}^{2n}$ with $T_p\mathbf{R}^{2n}$ by sending v to the tangent vector of $p + tv$; consequently, J_p^i acts on $\mathbf{R}^{2n}$ linearly. Let $\epsilon > 0$ be sufficiently small, let $p \in M, v \in T_pM$ satisfy $|p| < \epsilon$ and $|v - v_0| < \epsilon$. By transversality, $p + tJ_p^1 v$ and $p + tJ_p^2 v$ are in Ω^+ for $0 < t \leq 1$. Define

$$L = L(v, p) = \{p + sv \colon -2 < s < 2\} \subset M.$$

Let $e_1, \ldots, e_{2n-1}$ be the standard basis of $\mathbf{R}^{2n-1}$. We find an affine coordinate map ϕ on $\mathbf{R}^{2n}$ such that $\phi(p) = 0$, $\phi(p + v) = e_1$, and $\phi(p + v_j) = e_j$. We may also assume that the norms of ϕ and ϕ^{-1} have an upper bound independent of p, v. In what follows, all constants are independent of p, v. Proposition 3.6 (ii) will be used for this family of ϕ (with $p = 0$) depending on parameter v with v_0 to be chosen.

We want to apply Lemma 3.3 to $L(v, p)$. Here v, p are parameters and we suppress them in all expressions. For the above $L(p, v)$, we attach an approximate J-holomorphic curve u^1 of class $\mathcal{C}^{k+1+\alpha}$ such that

$$du^1(\partial_{\overline{z}}) = D^1(z) \cdot X^1(u^1(z)) + F^1(z)\overline{X^1(u^1(z))}, \tag{5.1}$$

$$|F^1(z)| \leq C|y|^{k+\alpha}, \quad (x, y) \in Q \overset{\text{def}}{=\!=} (-1, 1) \times (0, \epsilon).$$

We have an analogous u^2 on $\Omega^- \cup M$. We have

$$u^1(x, 0) = p + xv = u^2(x, 0) \text{ on } [-1, 1].$$

We know that $u(x, 0)$ is contained in $M \subset \overline{\Omega}_1^+ \cap \overline{\Omega}_1^-$ for $|x| < 1$. When $p = 0$ and $v = v_0$, we have $du^1(0)(\partial_x) = v_0$ and $du^1(0)(\partial_y) = J_0^1 du^1(0)(\partial_x) = J_0^1 v_0$ is contained in Ω_1^+, $-J_0^2 v_0$ is contained in Ω_1^- and both are transversal to M. Thus,

$$u^1(x, y) \in \Omega^+, \ (x, y) \in Q; \quad u^2(x, y) \in \Omega^-, \ (x, y) \in -Q. \tag{5.2}$$

The above hold for $v = v_0$ and $p = 0$. Since the derivatives of u are continuous in p, v, the above hold for $|p| < \epsilon$ and $|v - v'| < \epsilon$. And for a constant $C > 1$ independent of p, v,

$$\operatorname{dist}(u^i(x, y), M) \geq |y|/C, \quad (x, y) \in (-1)^{i-1}Q. \tag{5.3}$$

Step 1. Uniform bound of Fourier transform of f on transversal lines L in M. In this step and the next, we will assume that f is $\mathcal{C}^1$ on $\Omega_1 \cup \Omega_2$. We will verify this interior regularity in the final step.

Fix k. Recall from Step 0 that M is contained in $\mathbf{R}^{2n-1}$. Let v_0, ϵ be as in Step 0. By Lemma 3.5, there exist d vectors v_j in $\mathbf{R}^{2n-1}$ such that

$$(\partial_x, \partial_{y'})^\alpha = \sum_{1 \leq j \leq d} c_{\alpha,j}(v_j \cdot (\partial_x, \partial_{y'}))^{|\alpha|}, \quad 1 \leq |\alpha| \leq k. \tag{5.4}$$

Here $|v_j - v_0| < \epsilon$. Recall the line segment $L = \{p + sv_i \colon -1 \leq s \leq 1\}$ with $p \in M, |p| < \epsilon$. Fix such an L and denote its tangent vector v_i by v.

Note that when ϵ is sufficiently small, L has length $> |v_0|/2$. Let χ_0 be a cutoff function on M with compact support in $\Delta^{2n}_{|v_0|/(4n)} \cap M$. Then $\chi_0|_L$ has compact support. We will show that the Fourier transform of $\chi_0 f$ on L satisfies

$$(1 + |\xi|)^{k-1+\alpha-\beta} \left|\widehat{\chi_0 f|_L}(\xi)\right| < C_\beta \tag{5.5}$$

for all $\beta > 0$. Here C_β will be independent of p, $v_1, \dots, v_d$. We will verify (5.5) for $\xi = |\xi|v$, using $X_j^1 f = g_j^1$ on Ω^1 with $g_j^1 \in \mathcal{C}^k(\Omega^+ \cup M)$. For $\xi = -|\xi|v$, we use $X_j^2 f = g_j^2$ on Ω^2 with $g_j^2 \in \mathcal{C}^k(\Omega^- \cup M)$.

We now use approximate J holomorphic curves u^1, u^2 defined in Step 0. We drop the superscript in u^1, g_j^1, a_{jk}^1, etc.

Applying Whitney's extension (Lemma 2.3), we extend $\chi_0 \circ u(x, 0)$ to $\chi \in \mathcal{C}^\infty(Q)$ which has compact support in each $(-1, 1) \times \{y\}$. Moreover, $|\partial_{\overline{z}}\chi(x, y)| \leq C|y|^{k+\alpha}$. For brevity, denote $f \circ u, g_j \circ u, (\partial_{z_j} f) \circ u$ by f, g_j, h_j. Combining with (5.1), we get on Q

$$\begin{aligned} \partial_y f(x, y) &= i\partial_x f(x, y) - 2iD(x, y) \cdot g(u(x, y)) - 2iF(x, y) \cdot \overline{X(u)}f, \\ \partial_y \chi(x, y) &= i\partial_x \chi(x, y) + E(x, y). \end{aligned} \tag{5.6}$$

Here $(|E| + |F|)(x, y) \leq C|y|^{k+\alpha}$. And D, E, F are in $\mathcal{C}^{k+\alpha}(\overline{Q})$, and g is in $\mathcal{C}^k(\overline{Q})$.

In what follows, as required by (5.5) constants do not depend on L, p, v_j.

By (5.2), $u(x, y)$ is in Ω_1^+ for $|x| < 1$, $0 < y < \epsilon$. Define

$$\lambda(\xi, y) = \int_{\mathbf{R}} (\chi f)(x, y) e^{-i(x-iy)\xi}\, dx, \quad y \geq 0.$$

Note that $\widehat{\chi f|_L}(\xi) \equiv \lambda(\xi,0) = \lambda(\xi,\eta) - \int_0^\eta \partial_y\lambda(\xi,y)\,dy$. By (5.6), we obtain

$$\begin{aligned}\partial_y\lambda(\xi,y) &= \int_{\mathbf{R}} \big(i\partial_x(\chi f)(x,y) - (\chi f)(x,y)\xi\big)e^{-i(x-iy)\xi}\,dx \\ &\quad - 2i\int_{\mathbf{R}} (g(u)\cdot D\chi)(x,y)e^{-i(x-iy)\xi}\,dx + \int_{\mathbf{R}} (f(u)E)(x,y)e^{-i(x-iy)\xi}\,dx \\ &\quad - 2i\int_{\mathbf{R}} \chi F(x,y)\cdot(\overline{X}f)(u(x,y))e^{-i(x-iy)\xi}\,dx.\end{aligned}$$

By integration by parts, the first integral is zero. Since $g(u(x,y)), D(x,y) \in \mathcal{C}^k$ and $\eta\xi \geq 0$, the second, via integrating by parts k times, is less than $C(1+|\xi|)^{-k}$. The third is bounded by $C|E(x,y))| \leq Cy^{k+\alpha}$. We now estimate the last integral. This amounts to controlling the blow-up of derivatives of f at $u(x,y)$. By (5.3), Ω^+ contains $\Delta^{2n}_{y/C}(u(x,y))$. To apply Proposition 3.4 to the latter, we need a domain of fixed size. Let $\psi(\zeta) = u(x,y) + \zeta y/C$. So ψ^{-1} transforms J, X_j into $\hat{J}, \hat{X}_j = C^{-1}yd\psi^{-1}X_j$. On Δ^{2n}, we have

$$\hat{X}_j = \sum_{1\leq k\leq n}(b_{jk}\circ\psi\partial_{z_j} + a_{jk}\circ\psi\partial_{\overline{z}_j}).$$

Let $A' = (a_{jk}\circ\psi), B' = (b_{jk}\circ\psi)$. It is easy to see that on Δ^{2n}, $\inf\left|\begin{smallmatrix} B' & A' \\ \overline{A'} & \overline{B'}\end{smallmatrix}\right| \geq 1/C$ and $|(A',B')|_{k+1+\alpha} \leq C$ for some constant independent of v, p. Applying Proposition 3.4 to $\{\hat{X}_j\}$, we get a J-holomorphic curve $\hat{u}\colon \mathbb{D}_r \to \Delta^{2n}$ with $\hat{u}(0) = 0$, $d\hat{u}(0)\partial_\zeta = \partial_{x_m} - i\hat{J}_0\partial_{x_m}$. Here $r > 0$ is a constant independent of y. Then $\tilde{u}(\zeta) = \psi\circ\hat{u}(C\zeta/y)$ is J-holomorphic in J. We have $\tilde{u}\colon \Delta_{y/c} \to \Omega^+_1$, $\tilde{u}(0) = u(0)$, and

$$d\tilde{u}(0)(\partial_\zeta) = \partial_{x_m} - iJ_{u(0)}\partial_{x_m}.$$

So $d\tilde{u}(0)\partial_{\overline{\zeta}} = \partial_{x_m} + iJ_{u(0)}\partial_{x_m}$. A direct computation shows that the first and second order derivatives of $\tilde{u}$ are bounded by $C, C/y$, respectively. Since $X_jf = g_j$ and $d\tilde{u}(\partial_{\overline{\zeta}}) = \tilde{D}\cdot X(\tilde{u})$, then

$$\partial_{\overline{\zeta}}(f(\tilde{u}(\zeta))) = g(\tilde{u}(\zeta))\cdot\tilde{D}(\zeta).$$

Note that the derivative of $\tilde{D}$ is bounded by C/y. By the Cauchy-Green identity, we have

$$f(\tilde{u}(z)) = \frac{1}{2\pi i}\int_{|\zeta|=y/c}\frac{f(\tilde{u}(\zeta))}{\zeta - z}\,d\zeta + \frac{1}{\pi}\int_{|\zeta|<y/c}\frac{g(\tilde{u}(\zeta))\cdot\tilde{D}(\zeta)}{z-\zeta}\,d\xi d\eta.$$

At $z = 0$, derivatives of the first integral are bounded by C/y. Write $g(\tilde{u}(\zeta))\cdot\tilde{D}(\zeta)$ as $h_1(\zeta) + h_2(\zeta)$. Here $\mathcal{C}^1$ norms of h_1, h_2 are bounded by C/y, $h_1(\zeta) = 0$ on $|\zeta| < y/(4c)$, and $h_0(\zeta) = 0$ on $|\zeta| > y/(2c)$. The derivatives of the integral involving h_1 are bounded by C at $z = 0$. After applying translation $\zeta' = \zeta - z$, the integral involving h_0 has bounded derivatives at $z = 0$ too. We obtain $|\partial_{x^m}f(u(0))| = |\partial_{\overline{z}}f(u(0)) + \partial_z f(u(0))| \leq C/y$. Thus,

$$|\partial_y\lambda(\xi,y)| \leq C\left(y^{k-1+\alpha} + (1+|\xi|)^{-k}\right).$$

We also have $|\lambda(\xi,\eta)| \leq Ce^{-\eta\xi} \leq C_L|\eta\xi|^{-L}$. We may assume that $\xi \geq 1$. Choose $\eta = 1/(C|\xi|^\alpha)$. Then $\lambda(\xi,0) = \lambda(\xi,\eta) - \int_0^\eta \partial_y\lambda(\xi,y)\,dy$ satisfies

$$|\lambda(\xi,0)| \leq C(1+|\xi|)^{-(k+\alpha)} \tag{5.7}$$

for $\xi \geq 0$. Reason by using $X_j^2 f = g_j^2$ for $y_n \leq 0$, u^2, and by replacing e_n with $-e_n \in \mathbf{R}^n$. We get (5.7) for $\xi \leq 0$ and hence for $-\infty < \xi < \infty$.

By the Fourier inversion formula,

$$\chi f(p+xv) = \frac{1}{2\pi}\int_{\mathbf{R}} \lambda(\xi,0)e^{i\xi x}\,d\xi,$$
$$\partial_x^{k-1}(\chi f(p+xv)) = \frac{1}{2\pi}\int \lambda(\xi,0)(i\xi)^{k-1}e^{i\xi x}\,d\xi.$$

This shows that $\chi f(p+xv) \in \mathcal{C}^{k-1}$. By the mean-value-theorem and (5.7),

$$|\partial_x^{k-1}(\chi f)(p+x_2v) - \partial_x^{k-1}(\chi f)(p+x_1v)| \leq C\int \frac{|x_2-x_1|^{\alpha'}|\xi|^{\alpha'}}{(1+|\xi|)^{1+\alpha}}\,d\xi.$$

For any $\alpha' < \alpha$, we have $|(\chi f(p+\cdot v))|_{k-1+\alpha'} < C_{\alpha'}$. Therefore,

$$|\chi_0 f|_L|_{k-1+\alpha'} < C_{\alpha'},$$

where L is any line which is tangent to one of $v_1,\dots,v_d$ and passes through p for $p \in M$ near the origin. For the L, we can find an affine diffeomorphism R with $R(0) = 0 \in M$, sending Δ^{2n-1} into M, such that $R(\cdot,t)$ are lines parallel to L for $t \in \Delta^{2n-2}$. By Proposition 3.6 (ii) and hypothesis (5.4), we get $\partial^{k-2}(\chi_0 f) \in Lip\,(M)$.

Step 2. Uniform bound of derivatives of f on transversal J-holomorphic curves.

Fix k. By Lemma 3.5, there exist N vectors $v_j \in \mathbf{R}^{2n}$ with $|v_j| < 1$ such that

$$(\partial_x,\partial_y)^\alpha = \sum_{1\leq j\leq N} c_{\alpha,j}\big(v_j\cdot(\partial_x,\partial_y)\big)^{|\alpha|}, \quad 1 \leq |\alpha| \leq k. \tag{5.8}$$

By perturbing v_j, we may assume that $J_p(v_j\cdot(\partial_x,\partial_y))$ are not tangent to M at $p=0$ and hence in a neighborhood of $0 \in M$.

We are given vector fields $\{X_j\}$ defined on $\Omega^+ \cup M$. Recall that M is contained in $y_n = 0$ and Ω^+ is contained in $y_n > 0$. Applying Whitney's theorem via restriction and then extension, we may assume that $\{X_j\}$ is an almost complex structure defined in a neighborhood of M. We now apply Proposition 3.4 with the parameter set $P = \overline{\mathbb{D}}_{r_0}^{n-1}$. There are diffeomorphisms u_j, R_j of class $\mathcal{C}^{k+1+\alpha,k-1}$, which maps $\mathbb{D}_{r_0}\times\mathbb{D}_{r_0}^{n-1}$ into Ω with

$$du_j(0,t)(\partial_\xi) = v_j. \tag{5.9}$$

Moreover, $D_{j,r}(t) = u_j(\mathbb{D}_r,t) = R_j(\omega_{j,r}(t),t)$ satisfy

$$\mathbb{D}_{r/c_2} \subset \omega_{j,r}(t) \subset \mathbb{D}_{c_2 r}.$$

Also $R_j(0) = 0$, $u_j(0) = 0$, and

$$du_j(\cdot,t)(\partial_{\overline{\zeta}}) = J_0(v_j\cdot(\partial_x,\partial_y)) - i(v_j\cdot(\partial_x,\partial_y)).$$

We choose $r < r_0$ sufficiently small so that various compositions in u_j, R_j are well-defined. Also $\omega_{j,r}^+(t) = \omega_{j,r}(t)\cap\{y>0\}$ satisfies $R_j(\omega_{j,r}^+(t),t) \subset D_{j,r}^+(t) = D_{j,r}(t)\cap\Omega^+$. Write $(\tilde{D}_{j,r}^+(t),t) = u_j^{-1}(D_{j,r}^+(t))$. We apply the Cauchy-Green formula for $f\circ u_j(\cdot,t)$. Then

$$f(u_j(\tilde z,t)) = \frac{1}{2\pi i}\int_{\partial\tilde{D}_{j,r}^+(t)} \frac{f(u_j(\tilde\zeta,t))}{\tilde\zeta-\tilde z}\,d\tilde\zeta + \frac{1}{2\pi i}\int_{\tilde{D}_{j,r}^+(t)} \frac{\partial_{\overline{\tilde\zeta}} f(u_j(\tilde\zeta,t))}{\tilde\zeta-\tilde z}\,d\overline{\tilde\zeta}\wedge d\tilde\zeta.$$

Set $(\tilde{z}, t) = u_j^{-1} \circ R_j(z,t) = (\tau_j(z,t), t)$. By $du_j(\partial_{\overline{\zeta}}) = D(\zeta) \cdot X(u_j(\zeta))$ and $X_j f = g_j$, we get $\partial_{\overline{\zeta}} f(u_j) = D \cdot g$ and

$$
\begin{aligned}
f(R_j(z,t)) = \frac{1}{2\pi i} \int_{\zeta \in \partial\omega^+_{j,r}(t)} \frac{f(R_j(\zeta,t))}{\tau_j(\zeta,t) - \tau_j(z,t)}\, d\tau_j(\zeta,t) \\
+ \frac{1}{2\pi i} \int_{\omega^+_{j,r}(t)} \frac{g(R_j(\zeta,t)) \cdot D(\tau_j(\zeta,t),t)}{\tau_j(\zeta,t) - \tau_j(z,t)}\, d\overline{\tau_j(\zeta,t)} \wedge d\tau_j(\zeta,t).
\end{aligned}
$$

Recall that $R_j(\cdot,t)$ send $[-r/c, r/c]$ into $\partial D_j^+(t) \cap M$. Since $u \in \mathcal{C}^{k+1+\alpha,k-1}$, it is easy to see that $D(\tau_j(\zeta,t),t)$ are in $\mathcal{C}^{k+\alpha,0}$. Applying Lemma 4.1, we get $|f(R_j)|_{k-2+\beta,0} < C_\beta$ on D_{r/c_*} for any $\beta < 1$.

Step 3. Smoothness of f via families of J-holomorphic curves.

By the end of Step 2, we know that the $\mathcal{C}^{k-2+\beta}$ norms $f \circ R_j(\cdot,t)$ on $\overline{\mathbb{D}}_r^+$ are bounded in $t \in \mathbb{D}_r^{n-1}$ when r is small, and that $\partial_{t_1}^{k-2}(f \circ R_j(t))$ is continuous. Here $R_j(0) = 0 \in M$. Applying the Whitney extension, we extend $f(z)$ to Ω^- such that it has class $\mathcal{C}^{k-2+\beta}$ on $\Omega^- \cup M$. Then $f \circ R_j$ is of class $\mathcal{C}^{k-2+\beta}$ on $\mathbb{D}_r$. Therefore, for the extended function f, the $\mathcal{C}^{k-2+\beta}$ norms of $f(R_j(\cdot,t))$ on $\mathbb{D}_r$ are bounded and $\partial_\xi^{k-2}(f \circ R_j(\xi + i\eta, t))$ is continuous. We want to apply Proposition 3.6 (iii) to a family of diffeomorphisms $\tilde{R}_j \in \mathcal{C}^{k-1}$. This is achieved easily by taking $\tilde{R}_j(t_1, t_2, t') = u_j(t_1 + it_2, t')$ and treating $(t_2, t') \in \mathbf{R}^{2n-1}$ as parameters. By (5.8)-(5.9), we conclude $f \in \mathcal{C}^{k-3+\beta}(\Omega^+ \cup M)$ for all $\beta < 1$.

To finish the proof, we need to remove the assumption stated at the beginning of Step 1 that f is $\mathcal{C}^1$ on Ω_i. We also address the comment made after Theorem 5.1 on the interior regularity of f. Let $J = J^i \in \mathcal{C}^{k+\alpha}$ and $\Omega = \Omega_i$. Here we only need $k \geq 2$. Let v_j satisfy (5.8). By Proposition 3.4 we find a $\mathcal{C}^{k+1+\alpha,k-1}$ diffeomorphism $u_j(\zeta,t)$ defined in neighborhood of $0 \in \Omega$ such that $\zeta \to u_j(\zeta,t)$ is J-holomorphic for fixed $t \in \mathbb{D}_\epsilon^{n-1}$ and $du_j(0)(\partial_\xi) = v_j$. Drop the subscript j in u_j. Then u^{-1} defines a $\mathcal{C}^{k-1}$ coordinate system for a neighborhood of the origin. And u^{-1} transforms J into $\hat{J}$. Now $\mathbb{D}_\epsilon \times t$ are J-holomorphic discs in $\hat{J}$ for $|t| < \epsilon$ and ϵ small. Thus we can take $\hat{X}_1 = a(\zeta,t)\partial_{\overline{\zeta}} + b(\zeta,t)\partial_\zeta$ with $a, b \in \mathcal{C}^{k+\alpha,k-1}$ and t being parameters. Now $\hat{f} = f \circ u$ satisfies $\hat{X}_1 \hat{f} = \hat{g}_1 \in \mathcal{C}^{k,k-1}$. Here $\hat{X}_1 f = \hat{g}_1$ holds in the sense of distributions. We want to show that when restricted on $\mathbb{D}_\epsilon \times t$, $\hat{X}_1 \hat{f} = \hat{g}_1$ still holds as distributions. To verify it, fix a test function ϕ on $\mathbb{D}_\epsilon$ and take a sequence of test functions ϕ_i in $\mathbf{C}^{n-1}$ such that $\int_{\mathbf{C}^{n-1}} \phi_i = 1$ and $\operatorname{supp} \phi_i \subset B_{1/j}(t)$. Note that the formal adjoint $\hat{X}_1^*$ does not contain derivatives in t-variables and satisfies

$$
\int_{\mathbf{C}^n} \hat{g}_1 \phi \phi_i = \int_{\mathbf{C}^n} \hat{f} \hat{X}_1^*(\phi\phi_i) = \int_{\mathbf{C}^n} \hat{f} \phi_i \hat{X}_1^*(\phi).
$$

Since all functions in the integrands are continuous, letting i tend to ∞ yields

$$
\int_{\mathbf{C}} \hat{g}_1(\cdot,t)\phi = \int_{\mathbf{C}} \hat{f}(\cdot,t) \hat{X}_1^*(\phi).
$$

We have proved that $\hat{X}_1 \hat{f} = \hat{g}_1 \in \mathcal{C}^{k,k-1}$ in the sense of distributions and the coefficients of $\hat{X}_1$ are in $\mathcal{C}^{k+\alpha,k-1}$. Reasoning as at the end of Step 2 by the Cauchy-Green identity, we

see that $f \circ u_j \in \mathcal{C}^{k+\alpha,0}$. Now $f \circ u_j \in \mathcal{C}^{k-2,0}$, $u_j \in \mathcal{C}^{k-1}$ and Proposition 3.6 (iii) implies that $f \in \mathcal{C}^{k-3+\beta}$ for all $\beta < 1$. By $k \geq 4$, the proof of Theorem 5.1 is complete.

6. One-dimensional results

Throughout this section, Ω is a bounded open set in $\mathbf{C}$, and P is the closure of a bounded open set in a euclidean space. We assume that two points a, b in $\overline{\Omega} \times P$ can be connected by a smooth curve in $\overline{\Omega} \times P$ of length at most $C|b-a|$.

We start with the existence of isothermal coordinates with parameter. Recall that a diffeomorphism φ is said to transform a vector field X into $\tilde{X}$ if locally $d\varphi(X) = \mu\tilde{X}$. Denote by $\mathcal{C}^{k+\alpha,j}_{loc}(\Omega \cup \gamma, P)$ the set of functions which are in $\mathcal{C}^{k+\alpha,j}(K, P)$ for any compact subset K of $\Omega \cup \gamma$.

Proposition 6.1. *Let Ω be a domain in $\mathbf{C}$ and P be an open set in a euclidean space. Let $a \in \mathcal{C}^{k+\alpha,j}(\Omega, P)$ satisfy $|a|_{0,0} < 1$. If $x \in \Omega$ there exist a neighborhood U of x and a map $\varphi \in \mathcal{C}^{k+1+\alpha,j}(U, P)$ such that $\varphi(\cdot, t)$ are diffeomorphisms which map U onto their images and $\partial_{\overline{z}} + a(z,t)\partial_z$ into $\partial_{\overline{z}}$.*

Proof. Fix $x = 0 \in \Omega$. Let $\varphi(z,t) = z - a(0,t)\overline{z}$. Then $z \to \varphi(z,t)$ is invertible and transforms $\partial_{\overline{z}} + a(z,t)\partial_z$ into $\partial_{\overline{z}} + \tilde{a}(z,t)\partial_z$ with $\tilde{a}(0,t) = 0$. We still have $|\tilde{a}|_{\alpha,0} < \infty$. Let χ be a smooth function on $\mathbb{D}$ which has compact support and equals 1 on $\mathbb{D}_{1/2}$. Applying a dilation and replacing $\tilde{a}$ by $\chi\tilde{a} = b$ we achieve $|b|_{\alpha,0} < \epsilon_\alpha$ on $\mathbb{D}$ for ϵ_α in Lemma 2.2. Set $f = -(\mathrm{I} + Tb\partial_z)^{-1}Tb$. On $\mathbb{D}$ we have $f \in \mathcal{C}^{k+1+\alpha,j}$ and $|f|_{1,0} \leq C_\alpha |(\mathrm{I} + Tb\partial_z)^{-1}|_{1+\alpha,0}|b|_{\alpha,0}$. With the dilation for $\tilde{a}$, $|f|_{1,0}$ can be arbitrarily small. Therefore $z \to z + f(z,t)$ are indeed diffeomorphisms. Since $z + f(z,t)$ is annihilated by $\partial_{\overline{z}} + \tilde{a}\partial_z$, it transforms $\partial_{\overline{z}} + \tilde{a}(z,t)\partial_z$ into $\partial_{\overline{z}}$. $\square$

It is important that the above classical result (for the non-parameter case) allows one to interpret $\partial_{\overline{z}} f + a\partial_z f = g$ when a is merely $\mathcal{C}^\alpha$. Let $w = \varphi(z)$ be a local $\mathcal{C}^{1+\alpha}$ diffeomorphism such that $d\varphi(\partial_{\overline{z}} + a\partial_z) = \mu(w)\partial_{\overline{w}}$. Then $\partial_{\overline{z}} f + a\partial_z f = g$ holds in the w-coordinates, if $\partial_{\overline{w}}(f \circ \varphi^{-1}) = g \circ \varphi^{-1}(w)/\mu(w)$ holds in the sense of distributions. We restate Proposition 1.1 in a parameter version.

Proposition 6.2. *Let $0 < \alpha < 1$ and $k \geq j \geq 0$ be integers. Let γ be an embedded curve in $\mathbf{C}$ of class $\mathcal{C}^{k+1+\alpha}$. Let Ω_1, Ω_2 be disjoint open subsets of $\mathbf{C}$ such that both $\partial\Omega_1, \partial\Omega_2$ contain γ as relatively open subsets. Assume that $a_i \in \mathcal{C}^{k+\alpha,j}(\Omega_i \cup \gamma, P)$ satisfies $|a_i|_{0,0} < 1$ on $(\Omega_i \cup \gamma) \times P$. Let $f \in \mathcal{C}^{0,j}(\Omega_1 \cup \gamma \cup \Omega_2, P), b_i \in \mathcal{C}^{k+\alpha,j}(\Omega_i \cup \gamma, P)$ satisfy*

$$\partial_{\overline{z}} f + a_i \partial_z f = b_i \text{ on } \Omega_i, \qquad i = 1, 2.$$

Then $f \in \mathcal{C}^{k+1+\alpha,j}_{loc}(\Omega_i \cup \gamma, P)$.

Proof. As in the proof of Theorem 1.2 in section 5, we may assume that γ is the x-axis and $\Omega_1 = \mathbb{D}_r^+, \Omega_2 = \mathbb{D}_r^-$. In the following, all functions a_i, b_i, etc. are defined on Ω_i for some $r > 0$ and we will take smaller values for r for a few times. Apply the Whitney extension theorem with parameter (Lemma 2.3). We first find a function $\phi_i \in \mathcal{C}^{k+1+\alpha,j}$ with $\phi_i(\cdot, t) \in \mathcal{C}^2(\Omega_i \cup \gamma)$ such that $\phi_i(x, 0, t) = x$ and $\partial_{\overline{z}}\phi_i + a_i\partial_z\phi_i = O(|y|^{k+\alpha})$. Then ϕ_i sends $\partial_{\overline{z}} + a_i\partial_z$ into $\mu_i(\partial_{\overline{z}} + \tilde{a}_i\partial_z)$. Replace f, a_i by $f \circ \phi_i^{-1}, \tilde{a}_j$ on $\overline{\Omega}_i$. Therefore, we may assume

that $a_i(z,t) = O(|y|^{k+\alpha})$. Define $a = a_i$ on $\overline{\Omega}_i$. Then a is of class $\mathcal{C}^{k+\alpha,j}(\Omega_1 \cup \gamma \cup \Omega_2, P)$. Set $X = \partial_{\overline{z}} + a(z,t)\partial_z$.

Next, we find $g_i \in \mathcal{C}^{k+1+\alpha,j}$ on $\overline{\Omega}_i$ so that

$$Xg_i - b_i = O(|y|^{k+\alpha}), \quad g_i(x,0) = 0.$$

Replace f by $f - g_i$ on $\overline{\Omega}_i$. Therefore, we may assume that $b_i(z,t) = O(|y|^{k+\alpha})$. Define $b = b_i$ on $\overline{\Omega}_i$. Then b is of class $\mathcal{C}^{k+\alpha,j}(\Omega_1 \cup \gamma \cup \Omega_2, P)$.

By Proposition 6.1, there are diffeomorphisms $\psi(\cdot,t)$ with $\psi \in \mathcal{C}^{k+1+\alpha,j}(\mathbb{D}_r, P)$, which send X into $\mu\partial_{\overline{z}}$ with $\mu \in \mathcal{C}^{k+\alpha,j}(\mathbb{D}_r, P)$. Then $\partial_{\overline{z}}(f \circ \psi^{-1}) = b \circ \psi^{-1}/\mu$. Let $h = T_{\mathbb{D}_r}(b \circ \psi^{-1}/\mu)$ where r is sufficiently small. Then $h \in \mathcal{C}^{k+1+\alpha,j}$. Now $f \circ \psi^{-1} - h$ is holomorphic away from $\psi(\gamma)$, continuously up to the $\mathcal{C}^1$ curve $\psi(\gamma)$. Take a small disc D_r, independent of t and centered at $p \in \psi(\cdot,t_0)(\gamma)$. By the Cauchy formula, we express $f(\cdot,t)$ on Δ_r via the Cauchy transform on $\partial\Delta_r$ when t is in a small neighborhood of t_0. From $f \in \mathcal{C}^{0,j}$ and compactness of P we conclude $f \in \mathcal{C}^{k+1+\alpha,j}(\mathbb{D}_{r/2}, P)$. Recall that f is replaced by $f \circ \phi_i$. The original f is in $\mathcal{C}^{k+1+\alpha,j}_{loc}(\Omega_i \cup \gamma, P)$. □

Lemma 6.3. *Let $D \subset \mathbf{C}$ be a bounded domain with $\partial D \in \mathcal{C}^1$. Suppose that $v \in \mathcal{C}^1(D)$ and b are continuous functions on $\overline{D}$. Then v satisfies*

$$v + Tb = 0 \tag{6.1}$$

if and only if it satisfies

$$\partial_{\overline{z}} v + b = 0, \tag{6.2}$$

$$\mathcal{C}v = \frac{1}{2\pi i}\int_{\partial D} \frac{v(\zeta)}{\zeta - z}\, d\zeta = 0. \tag{6.3}$$

Here three identities are on D. Moreover, (6.3) holds on D if and only if v is the boundary value of a function that is holomorphic on $\mathbf{C} \setminus \overline{D}$, continuous on $\mathbf{C} \setminus D$, and vanishing at ∞.

Proof. Applying $\partial_{\overline{z}}$ to (6.1) gives us (6.2). On D, $\mathcal{C}v = v - T\partial_{\overline{z}}v$. Applying $T\partial_{\overline{z}}$ to (6.1) and using (6.1) again, we get $v - T\partial_{\overline{z}}v = 0$. Conversely, if v satisfies (6.2), then $v + Tb = v - T\partial_{\overline{z}}v = \mathcal{C}v$. The latter is zero by (6.3). Thus v satisfies (6.1).

It is a standard fact that when D is a bounded domain with $\mathcal{C}^1$ boundary and v is continuous on ∂D, then $\mathcal{C}v(z - tn(z)) - \mathcal{C}v(z + tn(z))$ converges to $v(z)$ uniformly on ∂D as $t \to 0^+$. Here n is the unit outer normal vector of ∂D. Then (6.3) implies that $\mathcal{C}v$ is continuous on $\mathbf{C} \setminus D$ and agrees with v on ∂D. That $\mathcal{C}v$ vanishes at ∞ is trivial. The converse follows from the Cauchy formula. □

Applying the above component-wise to the vector-valued functions, we get

Lemma 6.4. *Let $D \subset \mathbf{C}$ be a bounded domain with $\partial D \in \mathcal{C}^1$. Suppose that $v \in \mathcal{C}^1(D)$ and b are vectors of n continuous functions on $\overline{D}$.*

(*i*) *Let A be an $n \times n$ matrix of continuous functions on $\overline{D}$. Then v satisfies*

$$v + T(b + A\partial_z v) = 0 \tag{6.4}$$

if and only if v satisfies (6.3) and

$$\partial_{\overline{z}} v + b + A\partial_z v = 0. \tag{6.5}$$

(*ii*) *Let u be a continuous map from $\overline{D}$ into an open subset Ω of $\mathbf{C}^n$ with $v \in \mathcal{C}^1(D)$. Let $A \in \mathcal{C}^1(\Omega)$ be an $n \times n$ matrix. Then v satisfies*

$$v + T(b + A(v)\overline{\partial}_z v) = 0 \tag{6.6}$$

if and only if v satisfies (6.3) and

$$\partial_{\overline{z}} v + b + A(u)\overline{\partial_z v} = 0. \tag{6.7}$$

Here equations (6.3)-(6.7) are on D.

We prove a version of Theorem 1.3 with parameter.

Proposition 6.5. *Let $0 < \alpha < 1$ and let $k \geq j \geq 0$ be integers. Let Ω be a bounded domain in $\mathbf{C}$ with $\partial\Omega \in \mathcal{C}^{k+1+\alpha}$. Let $a, b \in \mathcal{C}^{k+\alpha,j}(\overline{\Omega}, P)$ be (scalar) functions satisfying $|a|_{\alpha,0} < \epsilon_\alpha$. Then*

$$v(\cdot,t) + T_\Omega b(\cdot,t) + T_\Omega(a(\cdot,t)\partial_z v(\cdot,t)) = 0 \tag{6.8}$$

has a unique solution $v(z,t)$ with $v \in \mathcal{C}^{k+\alpha+1,j}(\overline{\Omega}, P)$. Consequently,

$$\mathrm{I} + Ta\partial_z \colon \mathcal{C}^{k+1+\alpha,j}(\overline{\Omega}, P) \to \mathcal{C}^{k+1+\alpha,j}(\overline{\Omega}, P)$$

has a bounded inverse.

Proof. By Lemma 2.2, there exists a solution $v \in \mathcal{C}^{1+\alpha,j}$ to (6.8). The proposition is verified for $k = 0$. If $k \geq 1$, the assertion that $v \in \mathcal{C}^{k+1+\alpha,j}$ follows from Proposition 6.2 and Lemma 6.4. The last assertion in the proposition follows from $T_\Omega(\mathcal{C}^{k+\alpha,j}(\overline{\Omega}, P)) = \mathcal{C}^{k+1+\alpha,j}(\overline{\Omega}, P)$ and the open mapping theorem. $\square$

Remark 6.6. Let $0 < \alpha < 1$ and k, j be nonnegative integers. Let Ω be a bounded domain in $\mathbf{C}$ with $\partial\Omega \in \mathcal{C}^{k+1+\alpha}$. It would be interesting to know if

$$\mathrm{I} + Ta\partial_z \colon \hat{\mathcal{C}}^{k+1+\alpha,j}(\overline{\Omega}, P) \to \hat{\mathcal{C}}^{k+1+\alpha,j}(\overline{\Omega}, P)$$

has a bounded inverse, assuming $a \in \hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega}, P)$ and $\|a\|_{\alpha,0}$ is small. Lemma 2.2 is for the case when $a(\cdot,t)$ has compact support.

We now use the proof of Proposition 6.2 to study a problem in different directions. However, unlike the previous case, the next one fails in higher dimension.

Proposition 6.7. *Let $0 < \alpha < 1$ and $k \geq 0$ be an integer. Let γ be an embedded curve in $\mathbf{C}$ of class $\mathcal{C}^{k+1+\alpha}$. Let Ω_1, Ω_2 be disjoint open subsets of $\mathbf{C}$ such that both $\partial\Omega_1, \partial\Omega_2$ contain γ as relatively open subsets. Assume that $a_i \in \mathcal{C}^{k+\alpha}(\Omega_i \cup \gamma)$ satisfies $|a_i|_{0,0} < 1$ on $\Omega_i \cup \gamma$. Let E be an embedded $\mathcal{C}^1$ curve in $\mathbb{D}$ such that $\mathbb{D} \setminus E$ is open in $\mathbf{C}$ and has exactly two connected components ω_1, ω_2. Assume that u is a continuous map from $\mathbb{D}$ into $\Omega_1 \cup \gamma \cup \Omega_2$ such that $u \colon \omega_i \to \Omega_i$ are J-holomorphic with respect to $\partial_{\overline{z}} + a_i\partial_z$. Then E is a curve of class $\mathcal{C}^{k+1+\alpha}_{loc}$.*

Proof. The proof is a slight modification of the proof of Proposition 6.2. The problem is local. Fix $z_0 \in E$ and let $p = u(z_0)$. We may assume that near p, γ is contained in the real axis and Ω_1, Ω_2 are contained in the lower and upper half planes. Applying a local change of coordinates φ_i which is of class $\mathcal{C}^{k+1+\alpha}$ on $\Omega_i \cup \gamma$ and fixes γ pointwise, we may assume that $a_j(x,y) = O(|y|^{k+\alpha})$. Let a be a_i on $\Omega_i \cup \gamma$. Then $X = \partial_{\overline{z}} + a\partial_z$ is of class $\mathcal{C}^{k+\alpha}$ on

$\Omega_1 \cup \gamma \cup \Omega_2$. Near $p \in \gamma$, we apply a diffeomorphism ϕ of class $\mathcal{C}^{k+1+\alpha}$ which transforms X into $\partial_{\overline{z}}$. Let $g = \phi \circ \phi_i \circ u$ on $\omega_i \cup E$, which is holomorphic away from E. Since g is continuous and E is an embedded $\mathcal{C}^1$ curve, then g is holomorphic at z_0. It is easy to verify that g is biholomorphic near z_0. Consequently, E is of class $\mathcal{C}^{k+1+\alpha}$ near z_0. □

Example 6.8. Let E be an embedded $\mathcal{C}^1$ curve connecting $i, -i$ and dividing $\mathbb{D}$ into two components ω_1, ω_2. Let λ be a $\mathcal{C}^\infty$ function on $\mathbb{D}$ which is positive on ω_1 and negative on ω_2. The existence of such a function is trivial, by taking it vanishing to infinity order along E. We use the standard complex structure on $\mathbb{D} \times \mathbf{C} = \{\operatorname{Im} w < \lambda(z)\} \cup \{\operatorname{Im} w = \lambda(z)\} \cup \{\operatorname{Im} w > \lambda(z)\}$. Let $u(z) = (z, 0)$. Then $u\colon \omega_i \to \Omega_i$ are holomorphic, $\gamma = \{\operatorname{Im} w = \lambda(z)\}$ is $\mathcal{C}^\infty$, $u(\mathbb{D}) = \mathbb{D} \times 0$, but E needs not be $\mathcal{C}^\infty$.

The main purpose of next result is to provide another proof of Proposition 6.2. The proof does not yield a sharp result. To get the sharp result, we have to return to the argument in Proposition 6.2. We will deal with non-tangential boundary values. We will restrict to the non-parameter case.

Proposition 6.9. *Let $0 < \alpha < 1$ and $k \geq j \geq 0$ be integers. Let γ be an embedded curve in $\mathbf{C}$ of class $\mathcal{C}^{k+1+\alpha}$. Let Ω_1, Ω_2 be disjoint open subsets of $\mathbf{C}$ such that both $\partial\Omega_1, \partial\Omega_2$ contain γ as relatively open subsets. Assume that $a_i, b_i \in \mathcal{C}^{k+\alpha}(\Omega_i \cup \gamma)$ satisfy $|a_i|_{0,0} < 1$ on $\Omega_i \cup \gamma$. Suppose that $f|_{\Omega_i}$ are continuous and admit the same non-tangential boundary value function $f \in L^p(\gamma)$ with $p > 1$. Let f satisfy*

$$\partial_{\overline{z}} f + a_i \partial_z f = b_i \text{ on } \Omega_i, \qquad i = 1, 2. \tag{6.9}$$

Then $f \in \mathcal{C}^{k+1+\alpha}_{loc}(\Omega_i \cup \gamma)$.

Proof. We may assume that Ω_1, Ω_2 are two disjoint bounded simply connected domains whose boundaries are of class $\mathcal{C}^{k+1+\alpha}$. Apply a $\mathcal{C}^{k+1+\alpha}$ diffeomorphism ψ_i of $\overline{\Omega}_i$ onto $\overline{\Omega}'_i$ which transforms $\partial_{\overline{z}} + a_i\partial_z$ into $\partial_{\overline{z}}$. Such ψ_i exists in view of Proposition 6.1 by extending a_i to a neighborhood of γ via Whitney's extension theorem and by shrinking Ω at $p \in \gamma$. By a theorem of Kellogg, there exists a Riemann mapping $\phi_i \in \mathcal{C}^{k+1+\alpha}(\overline{\Omega'_i})$ which sends Ω'_i onto the upper half-plane. We may assume that $\gamma \neq \partial\Omega_j$ and γ is mapped into a compact subset by $\phi_j \circ \psi_j$.

Without loss of generality, we may assume that $\gamma = (-1, 1)$. We choose subdomain ω_j of Ω_j as follows: $\partial\omega_j$ contains $[-r_0, r_0]$; f has non-tangential limits at $r_0, -r_0 \in \gamma$; $\phi_j \circ \psi_i$ sends $\overline{\omega}_j$ onto $Q = [r', r''] \times [0, 1]$. Now, let ϕ be a Riemann mapping for Q. Note that ϕ is smooth on $\overline{Q}$ and $\phi' = 0$, $\phi'' \neq 0$ at vertices of Q. Let $\varphi_j = \phi \circ \phi_j \circ \psi_j$. Thus, $(\varphi_j^{-1})^* L^p(\partial\omega_j) \subset L^p(\partial\mathbb{D})$. (For our local results, we avoid the use of $\varphi_j^*(L^p(\partial\mathbb{D})) \subset L^q(\partial\omega_j)$ for $q < p/2$.)

Let $\mathcal{H}$ be the conjugate operator on $\partial\mathbb{D}$. Namely, for a real function $f \in L^p(\partial\mathbb{D})$ with $1 < p < \infty$, there is a holomorphic function h on $\mathbb{D}$ with $\operatorname{Im} h(0) = 0$ whose non-tangential boundary value is $f + i\mathcal{H}f$ with $\mathcal{H}f$ real-valued (Theorem 3.1, p. 57; Lemma 1.1, p. 103 in [3]). The $\mathcal{A}_i f = (\mathcal{H}(f \circ \varphi_i^{-1})) \circ \varphi_i$ is called the conjugate operator on $\partial\omega_j$ for $\partial_{\overline{z}} + a_i\partial_z$.

By a lemma of M. Riesz ([3], p. 113), $\|\mathcal{H}v\|_{L^p(\partial\mathbb{D})} \leq C_p\|v\|_{L^p(\partial\mathbb{D})}$ for $1 < p < \infty$. Thus, $\mathcal{H}^2 f = -f + c_f$ for $f \in L^p(\partial\mathbb{D})$ with c_f being a constant. By Privalov's theorem, $\mathcal{H}(L^p(\partial\mathbb{D}) \cap \mathcal{C}^{k+\alpha}(E)) \subset \mathcal{C}^{k+\alpha}_{loc}(E)$ for an arc E in $\mathbb{D}$.

From now on, we assume that $1 < p < \infty$. By our choice of ω_j, f is bounded on $\partial\omega_j \setminus \gamma$. Thus $f|_{\omega_j}$ has non-tangential limit functions in $L^p(\partial\omega_j)$. Recall that for $u \in L^p(\partial\mathbb{D})$,

$$\mathcal{H}u(z) = -\frac{1}{\pi}\, p.v. \int_{\partial\mathbb{D}} u(\zeta)\, d\log|\zeta - z|, \quad z \in \partial\mathbb{D}.$$

Thus for $u_i \in L^p(\partial\omega_j)$,

$$\begin{aligned} \mathcal{A}_i g(z) &= -\frac{1}{\pi}\, p.v. \int_{\partial\omega_i} u_i(\zeta)\, d\log|\varphi_i(\zeta) - \varphi_i(z)| \\ &\overset{\text{def}}{=\!=} -\frac{1}{\pi} \lim_{\epsilon\to 0} \int_{\partial\omega_i \cap \{|\varphi_i(\zeta)-\varphi_i(z)|>\epsilon\}} u_i(\zeta)\, d\log|\varphi_i(\zeta) - \varphi_i(z)|. \end{aligned} \tag{6.10}$$

Let $f_i \in \mathcal{C}^{k+1+\alpha}(\overline{\Omega}_i)$ be a solution to the inhomogeneous equation (6.9). We get

$$f = f_i + g_i, \quad \text{on } \overline{\Omega}_i, \quad \partial_{\overline{z}} g_i + a_i \partial_z g_i = 0. \tag{6.11}$$

Our assumption implies that $g_i|_{\omega_j}$ has non-tangential limits in $L^p(\partial\omega_j)$. By (6.10)-(6.11), we have

$$g_i = u_i + \sqrt{-1}\mathcal{A}_i u_i + ic_i.$$

Let χ_γ be the characteristic function of γ. Obviously, $E_i = \mathcal{A}_i((1-\chi_\gamma)u_i) \in \mathcal{C}^{k+1+\alpha}_{loc}(\gamma)$. On γ, we have $f_1 + g_1 = f_2 + g_2$ and hence

$$u_1 + \operatorname{Re} f_1 = u_2 + \operatorname{Re} f_2, \quad \mathcal{A}_1 u_1 + \operatorname{Im} f_1 + c_1 = \mathcal{A}_2 u_2 + \operatorname{Im} f_2 + c_2.$$

By the first identity, we obtain $\mathcal{A}_2(\chi_\gamma(u_2 - u_1)) \in \mathcal{C}^{k+1+\alpha}_{loc}(\gamma)$. The second shows

$$\mathcal{A}_2(\chi_\gamma u_1) - \mathcal{A}_1(\chi_\gamma u_1) \in \mathcal{C}^{k+1+\alpha}_{loc}(\gamma).$$

We assume that Ω_2 and γ have the same orientation. On γ, we rewrite (6.10) as

$$\mathcal{A}_1(\chi_\gamma u_1)(z) = \frac{1}{\pi}\, p.v. \int_\gamma u_1(\zeta)\, d\log|\varphi_1(\zeta) - \varphi_1(z)|,$$

$$\mathcal{A}_2(\chi_\gamma u_1)(z) = -\frac{1}{\pi}\, p.v. \int_\gamma u_1(\zeta)\, d\log|\varphi_2(\zeta) - \varphi_2(z)|.$$

Here the change of sign arises from the opposite orientation of γ in $\partial\omega_1$.

Assume now that $k \geq 1$. We may assume that $\gamma = (0,1)$. Therefore, on γ

$$(\mathcal{A}_1 - \mathcal{A}_2)(\chi_\gamma u_1)(x) = 2\mathcal{A}_1(\chi_\gamma u_1)(x) + E_3(x) + C(x),$$

where

$$E_3(x) = \frac{1}{\pi} \int_0^1 u_1(t)\, d\log\frac{|\varphi_2(t) - \varphi_2(x)|}{|\varphi_1(t) - \varphi_1(x)|} + C(x),$$

$$C(x) = \lim_{\epsilon\to 0} \left\{ \int_{I_2(x,e)} - \int_{I_1(x,e)} \right\} \frac{u_1(t)}{\pi}\, d\log|\varphi_1(t) - \varphi_1(x)|.$$

Here $I_j(x,\epsilon) = (0,1) \setminus (x - \epsilon_j, x + \epsilon'_j)$ with $e_j, e'_j > 0$, and

$$|\varphi_j(x - (-1)^j\epsilon_j) - \varphi_j(x)| = |\varphi_j(x + (-1)^j\epsilon'_j) - \varphi_j(x)| = c_j(x)^{-1}\epsilon.$$

Also, $c_j(x) > 0$ satisfies $|\varphi_j(t) - \varphi_j(x)| = c_j(x)|t - x + O(|t-x|^{1+\alpha})|$. Note that $e_j = e + O(\epsilon^{1+\alpha})$ and $e'_j = e + O(e^{1+\alpha})$. This shows that $(I_2(x,\epsilon) \setminus I_1(x,\epsilon)) \cup (I_1(x,\epsilon) \setminus I_2(x,\epsilon))$

is contained in $[e-Ce^{1+\alpha}, e+Ce^{1+\alpha}]\cup[-e-Ce^{1+\alpha}, -e+Ce^{1+\alpha}]$. Therefore, we can verify that

$$C(x)=\frac{1}{2\pi}\lim_{\epsilon\to 0}\left\{\int_{I_2(x,\epsilon)}-\int_{I_1(x,\epsilon)\}}\right\}\frac{u_1(t)}{x-t}\,dt=0$$

when x is a Lebesgue point of $u_1\in L^1_{loc}(\gamma)$. Hence, $C=0$ a.e. on γ. Since $E_3\in\mathcal{C}^{k-1+\alpha}_{loc}$, we obtain $2\mathcal{A}_1(\chi_\gamma u_1)\in\mathcal{C}^{k-1+\alpha}_{loc}(\gamma)$. Hence, $\mathcal{H}((\chi_\gamma u_1)\circ\varphi_1^{-1})\in L^p(\partial\mathbb{D})\cap\mathcal{C}^{k-1+\alpha}_{loc}(\tilde\gamma)$ for $\tilde\gamma=\varphi_1(\gamma)$. Since $(\chi_\gamma u_1)\circ\varphi_1\in L^p(\partial\mathbb{D})$, then $\mathcal{H}^2f=-f+c_f$ implies $(\chi_\gamma u_1)\circ\varphi_1^{-1}\in L^p(\partial\mathbb{D})\cap\mathcal{C}^{k-1+\alpha}_{loc}(\tilde\gamma)$. Therefore,

$$g_1\circ\varphi_1^{-1}|_{\partial\mathbb{D}}=u_1\circ\varphi_1^{-1}+i\mathcal{H}(u_1\circ\varphi_1^{-1})+ic_1\in L^p(\partial\mathbb{D})\cap\mathcal{C}^{k-1+\alpha}_{loc}(\tilde\gamma).$$

And $g_1\circ\varphi_1^{-1}\in\mathcal{C}^{k-1+\alpha}_{loc}(\mathbb{D}\cup\tilde\gamma)$. Hence $g_1\in\mathcal{C}^{k-1+\alpha}_{loc}(\omega_1\cup\gamma)$. Note that when $k\geq 2$, we can apply integration by parts and achieve $E_3\in\mathcal{C}^{k+\alpha}_{loc}$, and hence $g_1\in\mathcal{C}^{k+\alpha}_{loc}(\Omega_1\cup\gamma)$.

The above argument does not yield the sharp result. We now turn to the proof by using previous methods. We need to use a Fatou lemma. We assume that $\gamma=(-1,1)$. As used above, the g_j, given by (6.11) and holomorphic in $\partial_{\overline{z}}+a_j\partial_z$, has a non-tangential limit function in $L^p_{loc}(\gamma)$. We choose ω_j,φ_j as before. So $g_j\circ\varphi_j^{-1}$ is holomorphic on $\mathbb{D}$ with non-tangential limit function in $L^p(\partial\mathbb{D})$. By Fatou's lemma, $g_j\circ\varphi_j^{-1}(re^{i\theta})-g_j\circ\varphi_j^{-1}(e^{i\theta})$ and hence $f\circ\varphi_j^{-1}(re^{i\theta})-f\circ\varphi_j^{-1}(e^{i\theta})$ tend to zero in $L^p(\partial\mathbb{D})$ as $r\to 1^-$. Assume that $\gamma=(-1,1)$. Let r_0 be given in the definition of ω_j. Fix $0<r_1<r_0$. Write $[-r_1,r_1]=\{\varphi_j^{-1}(e^{i\theta});\theta_j\leq\theta\leq\theta_j'\}$. Let $\gamma_{j,t}=\{\varphi_j^{-1}(te^{i\theta})\colon\theta_j\leq\theta\leq\theta_j'\}$. Then $f|_{\gamma_{j,t}}$ tends to $f|_{\gamma_{j,1}}$ in L^p norm as $t\to 1^-$. More precisely, $\int_{\theta_j}^{\theta_j'}|f(\gamma_{2,t}(\theta))-f(\gamma_{2,1}(\theta))|^p\,d\theta$ tends to 0 as $t\to 1^-$. Note that $\gamma_{1,1},\gamma_{2,1}$ are the same set $[-r_1,r_1]$ with opposite orientations.

As in the proof of Proposition 6.2, taking smaller r_1 if necessary and using three changes of coordinates ϕ_1,ϕ_2,ψ and a solution of an inhomogeneous equation, we arrive at the case that a_j,b_j are zero. Now, f is holomorphic away from $\tilde\gamma=\psi(\gamma)$, $f|_{\gamma_{j,t}}$ tends to $f|_{\tilde\gamma_{j,1}}$ in L^p norm as $t\to 1^-$, and $\tilde\gamma_{1,1},\tilde\gamma_{2,1}$ are the same curve with opposite orientations. Applying the Cauchy formula to cancel boundary integrals in $\tilde\gamma_{1,1},\tilde\gamma_{2,1}$, we find the extension of f to a neighborhood of $p\in\tilde\gamma_{1,1}$ via a Cauchy transform on a small circle centered at p. Returning to the original coordinates, we obtain the desired conclusion for the original f. □

As mentioned in the introduction, our main result fails for harmonic functions. Let Ω be a bounded domain in $\mathbf{C}$ with $\partial\Omega\in\mathcal{C}^\infty$. Suppose that f is continuous on $\partial\Omega$ and dt is the arc-length element on $\partial\Omega$. Then $W_f(z)=\frac{1}{\pi}\int_{\partial\Omega}f(t)\log|\gamma(t)-z|\,dt$ is harmonic on $\mathbf{C}\setminus\partial\Omega$ and continuous on $\mathbf{C}$. However,

$$\partial_{n(s)}W_f=f(s)+\frac{1}{\pi}\int_{\partial\Omega}f(t)\partial_s\arg(\gamma(s)-\gamma(t))\,dt,$$
$$\partial_{-n(s)}W_f=f(s)-\frac{1}{\pi}\int_{\partial\Omega}f(t)\partial_s\arg(\gamma(s)-\gamma(t))\,dt.$$

Here $n(t)$ is the unit outer normal vector of $\partial\Omega$. In particular, if f is not smooth, then W_f cannot be smooth simultaneously on $\overline{\Omega}$ and $\mathbf{C}\setminus\Omega$.

We remark that if $W_f\in\mathcal{C}^1(\mathbf{C})$, then f and W_f must be zero.

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